



mathematical methods in chemical engineering

First semester 2025-2026

Madar Team

Lec(1)

* Differential equations:

- ordinary differential equations (ODEs)
- partial differential equations (PDEs)

* Variables:

- dependent
- independent



* $y = f(x)$, y : dependent on x
; x : independent

* $y = 3x^2 + x$

y changes according to x change

diff to time * $\frac{dy}{dx} + y = x$, y changes for x
 $\frac{dy}{dx}$ ind. (always!)

only one independent variable $\rightarrow$ (ODEs)

* $\frac{dT}{dt} = \alpha \frac{\partial^2 T}{\partial x^2}$

✓ 2 independent variables $\rightarrow$ (PDEs)

* System:

- $\frac{dy}{dx} + z = 3$
- $\frac{dz}{dx} + y = 6$

} two dep. variables
1 ind. variable $\rightarrow$ (ODEs)

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only one independent variable $\rightarrow$ (ODEs)

$$\text{* } \frac{dT}{dt} = \alpha \frac{\partial^2 T}{\partial x^2}$$

✓ 2 independent variables $\rightarrow$ (PDEs)

* system :

$$-\frac{dy}{dx} + z = 3$$

$$- \frac{dz}{dx} + y = 6$$

Two dep. Variables

1 ind. variable

→ (OPEs)

ODEs^k:



* 1st order

$$- \frac{dy}{dx} + 3 = e^x \rightarrow \boxed{\frac{dy}{dx} = f(x)}$$

$$- \frac{dy}{dx} + y = e^x \rightarrow \frac{dy}{dx} = f(x, y)$$

$$- \frac{dy}{dx} + y = 3 \rightarrow \frac{dy}{dx} = f(y)$$

* 2nd order^k

$$- \frac{d^2y}{dx^2} + f(x, y) \frac{dy}{dx} + g(x, y)y = a(x, y)$$

$$- \frac{d^2y}{dx^2} = b$$

$$- \frac{d^2y}{dx^2} + y = 12$$

$$- \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

* higher order

$$\frac{d^n y}{dx^n} + \frac{d^{n-1}y}{dx^{n-1}} \dots = a(x, y)$$

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x$$

* System of equations:

$$\left. \begin{aligned} \frac{dy}{dx} + yz &= x \\ \frac{dz}{dx} + y &= 3 \end{aligned} \right\} \text{solve both}$$



* constant coeff:

$$\alpha \frac{dy}{dx} + \beta y = \gamma, \quad \alpha, \beta, \gamma \text{ are constants}$$

$$3 \frac{dy}{dx} + 2y = 5$$

* non constant coeff:

$$x \frac{dy}{dx} + by = 2$$

* linear vs non linear (ODEs)

linear:

$$y'' - y = e^x$$

non linear:

$$(y')^2 = e^x \quad \text{Power}$$

$$yy'' - y = e^x \quad \text{the dep. var multiplied by a var.}$$

$$y' + y = e^y \quad \text{the dep. var. is not linear}$$

based on y dependent ←

Techniques of solution of 1st order (ODEs)

1) direct integration

$$y' = f(x) \rightarrow \frac{dy}{dx} = f(x) \rightarrow \int dy = \int f(x) dx$$
$$y = g(x)$$

$$y' = f(y) \Rightarrow \frac{dy}{dx} = f(y) \rightarrow \int \frac{dy}{f(y)} = \int dx$$

$$y' = f(x, y) \rightarrow \frac{dy}{dx} = f(x)g(y) \rightarrow \int \frac{dy}{g(y)} = \int \frac{dx}{f(x)}$$

example (1):

$$y' = e^{5x} \rightarrow \int dy = \int e^{5x} dx$$
$$y = \frac{e^{5x}}{5} + C$$



$$y' + y = 0 \rightarrow \frac{dy}{dx} + y = 0 \rightarrow \frac{dy}{dx} = -y$$

$$\int \frac{dy}{y} = \int -dx \rightarrow \ln y = -x + C$$

$$y = e^{-x+C}$$

$$y = e^{-x} \cdot e^C$$

$$y = C_2 e^{-x}$$

$$y' + y = 10 \rightarrow \frac{dy}{dx} + y = 10 \rightarrow \frac{dy}{dx} = 10 - y$$

$$\int \frac{dy}{10-y} = \int dx \rightarrow -\ln(10-y) = x + C_1$$

$$\ln(10-y) = -x + C_2$$

$$10-y = e^{-x} C_3$$

$$y = 10 - C_3 e^{-x}$$

* Separation of variables

$$\frac{dy}{dx} = f(x)g(y) \rightarrow \frac{dy}{g(y)} = \frac{dx}{f(x)}$$

$$\frac{dy}{dx} = \frac{x^2}{y^2} \rightarrow y^2 dy = x^2 dx, \frac{y^3}{3} = \frac{x^3}{3} + C_1$$

2] * exact solution of ODE



$$\underline{M(x, y)} + \underline{N(x, y)} \left(\frac{dy}{dx} \right) = 0$$

Example:

$$(2x^2y) + (xy^2 + 1) \frac{dy}{dx} = 0$$

$$\rightarrow \underline{M(x, y)} = 2x^2y$$

$$\rightarrow \underline{N(x, y)} = xy^2 + 1$$

* if $M_y = N_x$

$$\left(\frac{\partial M}{\partial y} \right) = \left(\frac{\partial N}{\partial x} \right) \rightarrow \text{exact ODE}$$

$$M_y \neq N_x \rightarrow \underline{\text{Not-exact ODE}}$$

1) direct integration.

2) exact Differential equation of (ODEs)

$$dy/dx = f(x,y) \rightarrow \underbrace{M(x,y)}_M \boxed{dx} + \underbrace{N(x,y)}_N \boxed{dy} = 0$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \leadsto 1^{st} \text{ ODEs is exact } *$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \leadsto 1^{st} \text{ ODEs is not exact } *$$

How to obtain solution for exact DE ::

a) integrate M with respect to x $\int M dx + \underline{h(y)}$ (constant of y)

b) differentiate [a] w.r.t y

c) equate [b] w.r.t N

d) $\int h'(y) dy \leadsto$ solution in [a]



example section ::

$$\rightarrow \underbrace{(y \cos x + 2x e^y)}_M + \underbrace{(\sin x + x^2 e^y + 2)}_N \frac{dy}{dx} = 0$$

$$\frac{\partial M}{\partial y} = \cos x + 2x e^y \quad \left. \vphantom{\frac{\partial M}{\partial y}} \right\} \text{(exact)}$$

$$\frac{\partial N}{\partial x} = \cos x + 2x e^y$$

$$\begin{aligned} 1) \int (y \cos x + 2x e^y) dx & \quad \left\{ \begin{array}{l} 2) \sin x + x^2 e^y + h'(y) \\ 3) \sin x + x^2 e^y + h(y) = N \end{array} \right. \\ = y \sin x + x^2 e^y + h(y) \end{aligned}$$

Solu ::

$$h(y) = 2 \leadsto h(y) = 2y \leadsto (y \sin x + x^2 e^y + 2y)$$

$$\rightarrow (2xy + 9x^2) + (2y + x^2 + 1)y' = 0$$

$$a) \frac{\partial M}{\partial y} = 2x, \frac{\partial N}{\partial x} = 2x \rightarrow \text{exact}$$

$$1) \int (2xy + 9x^2) dx = x^2y + 3x^3 + h(y)$$

$$2) x^2 + h'(y)$$

$$3) x^2 + h'(y) = 2y + x^2 + 1$$

$$h'(y) = 2y + 1$$

$$4) h(y) = y^2 + y$$

$$5) x^2y + 3x^3 + y^2 + y$$

$$\rightarrow \frac{dy}{dx} = -\frac{(x^3 - 3xy^2)}{3x^2y + y^3}$$

$$(3x^2y + y^3) dy = -(x^3 + 3xy^2) dx$$

$$\int (3x^2y + y^3) dy + \int (x^3 + 3xy^2) dx = 0$$

$$\frac{\partial M}{\partial y} = 3xy, \frac{\partial N}{\partial x} = 3xy \rightarrow \text{exact}$$

3) integrating factor:-

$$dy/dx + P(x)y = Q(x) \rightarrow (\text{any Poly})$$

$$- \mu = e^{\int P(x) dx}$$

$$- y = \frac{1}{\mu} \int \mu Q(x) dx$$

example section:-

$$\rightarrow dy/dx + 2y = e^{-2x}$$

$$P(x) = 2, Q(x) = e^{-2x}$$

$$\mu = e^{\int P(x) dx}$$

$$= e^{2x}$$

$$\rightarrow y = \frac{1}{e^{2x}} \int e^{2x} e^{-2x} dx$$

$$= \frac{1}{e^{2x}} \int dx$$

$$= e^{-2x}(x + C)$$

$$\rightarrow dy/dx + xy = \frac{1}{x}$$

$$P(x) = x, Q(x) = \frac{1}{x}$$

$$\rightarrow \mu = e^{\int x dx} = e^{x^2/2}$$

$$\rightarrow y = \frac{1}{e^{x^2/2}} \int \frac{e^{x^2/2}}{x} dx$$

$$u = x^2/2$$

$$du = x dx$$

$$dx = \frac{du}{x}$$

$$= e^{-x^2/2} \times \frac{Ei(u)}{2} + C$$

4) non-exact equations :-

Three cases :-

* Case (1) :-

i.e. $\mu = M(X) \rightarrow$ only

$$\ln \mu - \ln c = - \int \frac{\mu X - M Y}{\mu} dx$$

* we will multiply μ * (function) $\rightarrow$ exact $\rightarrow$ solution

* Case (2) :-

$\mu = M(Y)$ only

$$\ln \mu = c \int \frac{\mu X - M Y}{\mu} dy$$

* Case (3) :-

$\mu = M(X, Y)$

Partial differential equation

Note:

the μ the integrating factor converts an non exact eq. to exact.

example :-

$$\rightarrow (3xy + y^2) + (x^2 + xy) \frac{dy}{dx} = 0$$

$$\frac{\partial M}{\partial y} = 3x + 2y \quad \frac{\partial N}{\partial x} = 2x + y$$

non exact

$$\frac{\partial \mu}{\partial x} = 2x + y$$

diff

$$\ln \mu = - \int \frac{\mu x - M y^2}{\mu} dx$$

$$= - \int \frac{(2x^2 + y) - (3x + 2y)}{x^2 + xy} dx$$

$$x^2 + xy$$

$$\frac{\mu}{x} = x$$

$$[\mu = x] \text{ (i.e.)}$$

$$\leftarrow \ln \mu = \int \frac{x+y}{x(x+y)} dx = \ln|x|$$

$$\mu * ((3xy + y^2) + (x^2 + xy) \frac{dy}{dx} = 0)$$

$$(3x^2y + xy^2) + (x^3 + x^2y) \frac{dy}{dx} = 0$$

$$\frac{\partial M}{\partial x} = 3x^2 + 2xy, \quad \frac{\partial N}{\partial y} = 3x^2 + 2xy$$

exact solution

5) Bernoulli equation :

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

$$\mu(x) = e^{\int P(x) dx}$$

$$v = \frac{1}{\mu} \int \mu Q(x) dx$$

$$y = v^{-1}$$

$$P(x) = (1-n)P(x)$$

$$Q(x) = (1-n)Q(x)$$

example :-

$$\rightarrow y' + \frac{y}{x} = x^3 y^2$$

$$P(x) = \frac{1}{x}, \quad Q(x) = x^3$$

$$\mu(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln|x|} = x^{-1}$$

$$v = x^u \int (x^{-u} - x^3) dx$$

$$= x^u \int -x^{-1} dx$$

$$= -x^u \ln(x) + C$$

$$y = \frac{-1}{x^u \ln(x)} + C$$

6) Riccati equation :-

$$dy/dx = P(x)y^2 + Q(x)y + R(x)$$

Special case :-

$$P(x) = -1$$

We assume $y = \frac{1}{u} \frac{du}{dx}$

So many steps we get to :-

$$u'' - Q(x)u' - R(x)u = 0 \quad *$$

When we have constant coeff.

$$u = e^{rx}, u' = r e^{rx}, u'' = r^2 e^{rx}$$

$$u = c_1 e^{rx} + c_2 e^{rx}$$

$$\frac{du}{dx} = \dots$$

$$y = \frac{du}{dx} / u$$



Example :-

$$\rightarrow dy/dx + y^2 = y + 6$$

$$P(x) = -1, Q(x) = 1, R(x) = 6$$

assume $y = \frac{1}{u} \frac{du}{dx}$

$$u = e^{rx}$$

$$u' = r e^{rx}$$

$$u'' = r^2 e^{rx}$$

$$y = \frac{-2c_1 e^{-2x} + 3c_2 e^{3x}}{c_1 e^{-2x} + c_2 e^{3x}} \quad *$$

$$u'' - Q(x)u' - R(x)u = 0$$

$$r^2 e^{rx} - r e^{rx} - 6 e^{rx} = 0$$

$$(r+2)(r-3)$$

$$r = -2 \quad \left| \begin{array}{l} u = c_1 e^{-2x} + c_2 e^{3x} \\ \frac{du}{dx} = -2c_1 e^{-2x} + 3c_2 e^{3x} \end{array} \right. \rightarrow$$

7) linear equation coefficients

$$(ax + by + c)dx + (\alpha x + \beta y + \gamma)dy = 0$$

$$* \ln x = - \int \frac{\alpha + \beta y}{\beta y^2 + (b + \alpha)y + a} dy + c$$

$$* u = y/x$$

example :-

$$\rightarrow (x + 2y)dx + x dy = 0$$

$$\rightarrow (x + 2y + 0)dx + (x + 0y + 0)dy = 0$$

$$\frac{\partial u}{\partial y} = 2 \quad (\text{non-exact})$$

$$\frac{\partial u}{\partial x} = 1$$

$$\ln x = - \int \frac{10 + \phi u}{0 + (2+1)u + 1} du$$

$$= - \int \frac{1}{3u + 1} du$$

$$\ln x = \frac{-2 \ln(3u + 1)}{3} + c$$



$$3 \ln x = - \ln(3u + 1) + c_1$$

$$\ln(x^3 (3u + 1)) = c_1$$

$$x^3 (3 \frac{y}{x} + 1) = c_2$$

$$3yx^2 + x^3 = c_2$$

$$y = \frac{c_2 - x^3}{3x^2}$$

8) first order of 2nd type

$$(dy/dx)^2 - \alpha dy/dx + \beta y = g(x)$$

• assume $P = dy/dx$

$$P^2 - \alpha P + \beta y = g(x)$$

$\nearrow dy/dx$

$$P(x) = \frac{\alpha \pm \sqrt{\alpha^2 - 4(\beta y - g(x))}}{2}$$

• $u(x) = \text{طراز الحذر}$

$$\bullet \quad du/dx = \square$$

(+) $\text{نحوه و طراز در معادله}$ (-) $\text{نحوه و طراز در معادله}$

$\rightarrow$ example :-

$$(dy/dx)^2 - 2 dy/dx + y = x - 1$$

$$P = dy/dx$$

$$P^2 - 2P + y = x - 1 \quad \alpha = 2, \beta = 1$$

$$P = \frac{2 \pm \sqrt{2^2 - 4(y - x + 1)}}{2}$$

$$dy/dx = 1 \pm \sqrt{-y + x}$$

$$u = x - y$$

$$(du/dx = dx - dy) \cdot x dx$$

$$\frac{du}{dx} = 1 - \frac{dy}{dx}$$

$$du/dx = 1 - (1 \pm \sqrt{u})$$

$$= \mp \sqrt{u}$$



$$(+) \quad du/dx = +\sqrt{u}$$

$$\int \frac{du}{\sqrt{u}} = \int dx$$

$$2\sqrt{u} = x + c$$

$$4u = (x + c)^2$$

$$u = \frac{(x + c)^2}{4}$$

$$x - y = (x + c)^2 / 4$$

$$y = x - (x + c)^2 / 4$$

(-)

$$du/dx = -\sqrt{u}$$

$$\int \frac{du}{\sqrt{u}} = \int -dx$$

$$2\sqrt{u} = -x + c$$

$$4u = (-x + c)^2$$

$$x - y = (-x + c)^2 / 4$$

$$y = x - (-x + c)^2 / 4$$



9) Solution of system of homogeneous linear (ODEs)

$$\bullet \frac{dy_1}{dx} = a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + \dots$$

$$\bullet \frac{dy_2}{dx} = a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + \dots$$

$$\bullet \frac{dy_3}{dx} = a_{31}y_1 + a_{32}y_2 + a_{33}y_3 + \dots$$

$$\rightarrow \frac{dy}{dx} = Ay$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



$$\bullet y(x) = C_1 U_1 e^{\lambda_1 x} + C_2 U_2 e^{\lambda_2 x} + C_3 U_3 e^{\lambda_3 x} \dots$$

C_1, C_2, C_3 constants. / $\lambda_1, \lambda_2, \lambda_3$ eigen values

$$U_1 = \begin{bmatrix} U_{11} \\ U_{21} \\ U_{31} \end{bmatrix}, U_2 = \begin{bmatrix} U_{21} \\ U_{22} \\ U_{23} \end{bmatrix}, U_3 = \begin{bmatrix} U_{31} \\ U_{32} \\ U_{33} \end{bmatrix}$$

↓
eigen vectors

* eigen value and eigen vector

$AX=0$, $A \rightarrow$ matrix of value - $X \rightarrow$ matrix of vectors

example :-

$$\bullet 3x_1 + x_2 = 0$$

$$\bullet 2x_1 + 4x_2 = 0$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

• Matrices :-

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{matrix} \text{Row} \\ \text{column} \end{matrix} \rightarrow \begin{matrix} 2 \times 2 \\ (R) \quad (C) \end{matrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{matrix} 3 \times 3 \\ 1 \quad 2 \quad 3 \end{matrix}$$

• Vectors :-

$$X = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightarrow \begin{matrix} 2 \times 1 \\ (C) \quad (R) \end{matrix}$$

$$X = \begin{bmatrix} 5 & 8 \end{bmatrix} \rightarrow \begin{matrix} 1 \times 2 \end{matrix}$$

* matrix become vector if Row, column = 1

- 1 Row and n columns $\rightarrow$ Row vector
- 1 column and n Rows $\rightarrow$ column vector

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\Rightarrow A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$$

$$\Rightarrow A+B = \begin{bmatrix} 6 & 12 \\ 8 & 10 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 5 \\ 2 & 2 \end{bmatrix}$$



• if $A(2 \times 2)$, $B(2 \times 1) \rightarrow$ you can't find $(A+B)$

we must have same number of rows and columns.

* multiplication :-

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} * \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \end{bmatrix}$$

ex:-

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} * \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} (1*5 + 3*7) & (1*6 + 3*8) \\ (2*5 + 4*7) & (2*6 + 4*8) \end{bmatrix} = \begin{bmatrix} 26 & 30 \\ 38 & 44 \end{bmatrix}$$

• Matrix $\times$ vector

$$[2 \times 3] [3 \times 1] = [2 \times 1]$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (a_{11}x_1 + a_{12}x_2 + a_{13}x_3) \\ (a_{21}x_1 + a_{22}x_2 + a_{23}x_3) \end{bmatrix}$$

$$2 \times 3 \quad 3 \times 1$$

$\rightarrow$ Back to eigen value and eigen vector.

$$AX = 0 \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow a_{11}x_1 + a_{12}x_2 = 0$$

$$a_{21}x_1 + a_{22}x_2 = 0$$

$$\sim AX = \lambda I X \quad \lambda \text{ constant, } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ identity matrix}$$

$$AX - \lambda I X = 0$$

$$(A - \lambda I)X = 0$$

$$A - \lambda I = 0$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$AX = \lambda I X = 0$$

$$\lambda x_1 + 0x_2 = 0$$

$$0x_1 + \lambda x_2 = 0$$

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$I \lambda X = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

نتيجة $\rightarrow$

$$(a_{11} - \lambda)(a_{22} - \lambda) - (a_{12})(a_{21}) = 0$$

$$a_{11}a_{22} - a_{11}\lambda - a_{12}\lambda + \lambda^2 - a_{12}a_{21} = 0$$

$$a\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

$$\lambda = \frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2a_{11}}$$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

(eigen values)

Example:-

$$\rightarrow AX = 0$$

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{let } \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda + 1)(\lambda - 5)$$

eigen value *

to get eigen vectors:-
- Start with $\lambda = -1$

$$\begin{bmatrix} 1 - (-1) & 2 \\ 4 & 3 - (-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 + 2x_2 = 0$$

$$4x_1 + 4x_2 = 0$$

$$x_1 = -x_2, \quad x_1 = -x_2$$

assume

$$x_1 = 1, \quad x_2 = -1$$

$$x_1 = 2, \quad x_2 = -2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(eigen vectors) when $\lambda = -1$

when $\lambda = 5$

$$\begin{bmatrix} 1 - (5) & 2 \\ 4 & 3 - (5) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4x_1 + 2x_2 = 0$$

$$4x_1 - 2x_2 = 0$$

$$x_2 = 2x_1$$

$$\text{assume } x_1 = 1 \Rightarrow x_2 = 2$$

$$x_1 = 5 \Rightarrow x_2 = 10$$

(eigen vector)

Solution of system of (ODEs)

① $dy_1/dx = a_{11}y_1 + a_{12}y_2$ $dy_2/dx = k_y$
 $dy_2/dx = a_{21}y_1 + a_{22}y_2$ (general form)

$\Rightarrow \int dy/y = \int k dx \rightarrow \ln y = kx + c \rightarrow y = Ce^{kx}$

② $dy_1/dx = \lambda y_1 \Rightarrow y_1 = Ae^{\lambda_1 x}$
 $dy_2/dx = \lambda_2 y_2 \Rightarrow y_2 = Be^{\lambda_2 x}$

② in ①

$\lambda_1 Ae^{\lambda_1 x} = a_{11}Ae^{\lambda_1 x} + a_{12}Be^{\lambda_2 x}$
 $\lambda_2 Be^{\lambda_2 x} = a_{21}Ae^{\lambda_1 x} + a_{22}Be^{\lambda_2 x}$

$\Rightarrow \lambda A = a_{11}A + a_{12}B$ because $e \neq 0$

$\lambda B = a_{21}A + a_{22}B$ $\rightarrow$ because $e \neq 0$

Let $\lambda_1 = \lambda_2 = \lambda \Rightarrow A\lambda = a_{11}A + a_{12}B$

$B\lambda = a_{21}A + a_{22}B$

$\rightarrow (a_{11} - \lambda)A + a_{12}B = 0$

$a_{21}A + (a_{22} - \lambda)B = 0$

$\rightarrow \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Let: $\begin{bmatrix} \dots \\ \dots \end{bmatrix} = 0 \Rightarrow \lambda_1, \lambda_2$ Eigen values



substitute for each $\lambda_1, \lambda_2 \rightarrow$ gives A, B

$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

$u_1 = \alpha \begin{bmatrix} A \\ B \end{bmatrix}, u_2 = \beta \begin{bmatrix} A \\ B \end{bmatrix}$

example:-

solve:- $dy_1/dx = 6y_1 - 3y_2$

$dy_2/dx = 2y_1 + y_2$

$A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}, A - \lambda I = \begin{bmatrix} 6 - \lambda & -3 \\ 2 & 1 - \lambda \end{bmatrix}$

$\begin{bmatrix} 6 - \lambda & -3 \\ 2 & 1 - \lambda \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Let: $(6 - \lambda)(1 - \lambda) + 6 = 0$

$6 - 6\lambda - \lambda + \lambda^2 + 6 = 0$

$\lambda^2 - 7\lambda + 12 = 0$

$(\lambda - 3)(\lambda - 4) = 0$

$\lambda = 3, \lambda = 4$

$\rightarrow$ e^{3x}



when $\lambda = 4$: $\begin{bmatrix} 6-4 & -3 \\ 2 & 1-4 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2A - 3B = 0 \rightarrow B = 2/3A$
 $2A - 3B = 0 \rightarrow B = 2/3A$

$\rightarrow U_1 = \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

When $\lambda = 3$: $\begin{bmatrix} 6-3 & -3 \\ 2 & 1-3 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3A - 3B = 0 \rightarrow A = B$
 $2A - 2B = 0 \rightarrow A = B \rightarrow U_2 = \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$y = C_1 \alpha \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4x} + C_2 \beta \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3x}$

$\rightarrow y_1 = 3C_1 \alpha e^{4x} + C_2 \beta e^{3x}$
 $= 3C_{11} e^{4x} + C_{22} e^{3x}$

$\rightarrow y_2 = 2C_1 \alpha e^{4x} + C_2 \beta e^{3x}$
 $= 2C_{11} e^{4x} + C_{22} e^{3x}$



*eigen values $\Rightarrow$ real + distinct $\{ \lambda_1, \lambda_2 \rightarrow \lambda_1 \neq \lambda_2 \}$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{\lambda_1 x} + C_2 \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e^{\lambda_2 x}$

$\Rightarrow$ Real + repeated $\{ \lambda_1 = \lambda_2 = \lambda \text{ (real)} \}$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} e^{\lambda x} + C_2 \left(x \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \right) e^{\lambda x}$

$\Rightarrow$ complex roots $\{ \lambda_{1,2} = \alpha \pm \beta i \}$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \left\{ C_1 \left(\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \cos \beta x - \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \sin \beta x \right) + \right.$

$\left. C_2 \left(\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \sin \beta x + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \cos \beta x \right) \right\} e^{\alpha x}$

example:- Repeated roots

$y_1' = 3y_1 + 18y_2$

$y_2' = 3y_2$



$\begin{bmatrix} dy_1/dx \\ dy_2/dx \end{bmatrix} = \begin{bmatrix} 3 & 18 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

eigen values:- $A - \lambda I$

let $\begin{bmatrix} 3-\lambda & 18 \\ 0 & 3-\lambda \end{bmatrix} = 0 \Rightarrow (3-\lambda)^2 - 18 \times 0 = 0$
 $\Rightarrow (3-\lambda)^2 = 0 \rightarrow \lambda = 3, 3$

1st eigenvector. $\begin{bmatrix} 0 & 13 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$0v_1 + 13v_2 = 0 \Rightarrow v_2 = 0$$

$$v_1 = 1 \rightarrow v_2 = 0 \Rightarrow v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$v_1 = 10 \rightarrow v_2 = 0$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3x} \quad \text{first soln.}$$

• Solve $(A - 3I)w = 0$

$$\rightarrow \begin{bmatrix} 3-3 & 13 \\ 0 & 3-3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 13 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow 0w_1 + 13w_2 = 0 \rightarrow w_2 = 0$$

$$\rightarrow w = \begin{bmatrix} 0 \\ 1/13 \end{bmatrix} = c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\rightarrow y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = (c_1 x + c_2) e^{3x} =$$

$$c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3x}$$

$$\rightarrow y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3x} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3x}$$

case 3 :-

example :-

$$\frac{dy_1}{dx} = 2y_1 - 5y_2 \Rightarrow \begin{bmatrix} dy_1/dx \\ dy_2/dx \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$A - \lambda I \rightarrow \text{let } \begin{bmatrix} 2-\lambda & -5 \\ 1 & 2-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)^2 + 5 = 0 \rightarrow 4 - 2\lambda - 2\lambda + \lambda^2 + 5 = 0$$

$$\lambda^2 - 4\lambda + 9 = 0 \rightarrow \lambda = 2 \pm i\sqrt{5} \quad \text{complex Root}$$

$$\alpha = 2, \beta = \sqrt{5}$$

$$\textcircled{1} \Rightarrow \begin{bmatrix} 2-(2+i\sqrt{5}) & -5 \\ 1 & 2-(2+i\sqrt{5}) \end{bmatrix} = \begin{bmatrix} -i\sqrt{5} & -5 \\ 1 & -i\sqrt{5} \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow -i\sqrt{5}v_1 - 5v_2 = 0 \rightarrow v_2 = -i\sqrt{5}/5 v_1$$

$$\rightarrow v_1 - i\sqrt{5}v_2 = 0 \rightarrow v_2 = i/\sqrt{5} v_1$$

$$v = \begin{bmatrix} 1 \\ +1/\sqrt{5} i \end{bmatrix} = \begin{bmatrix} 1+0i \\ 0+1/\sqrt{5} i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/\sqrt{5} \end{bmatrix} i$$

$$\rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{2x} \left\{ c_1 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos\sqrt{5}x - \begin{bmatrix} 0 \\ 1/\sqrt{5} \end{bmatrix} \sin\sqrt{5}x \right) + \right.$$

$$c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin\sqrt{5}x - \begin{bmatrix} 0 \\ 1/\sqrt{5} \end{bmatrix} \cos\sqrt{5}x \right) \}$$

chapter(2)

Solution of 2nd order ODEs

$$y'' + P(x)y' + Q(x)y = g(x)$$

method (1) :

$$y'' + a_1 y' + a_0 y = g(x)$$

constant coeff. 2nd order ODEs.

$$\Rightarrow y(x) = y_c(x) + y_p(x)$$

• $y_c(x)$? $\Rightarrow y'' + a_1 y' + a_0 y = 0$

assume

$$y_c = e^{rx}, y'_c = r e^{rx}, y''_c = r^2 e^{rx}$$

$$\Rightarrow r^2 e^{rx} + a_1 r e^{rx} + a_0 e^{rx} = 0$$

$$e^{rx} (r^2 + a_1 r + a_0) = 0$$

$$r^2 + a_1 r + a_0 = 0 \rightarrow r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

$$r_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2}, r_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2}$$

$$\Rightarrow y_c(x) = A e^{r_1 x} + B e^{r_2 x}$$

case (1) : $r_1 \neq r_2$ (real roots)

$$y_c(x) = A e^{r_1 x} + B e^{r_2 x}$$

case (2) : $r_1 = r_2 = r$ (real roots)

$$y_c(x) = (A + Bx) e^{rx}$$

case (3) : $r_1 \neq r_2$ (complex) $r_1 = \alpha + \beta i, r_2 = \alpha - \beta i$

$$y_c(x) = (A \cos \beta x + B \sin \beta x) e^{\alpha x}$$

• $y_p(x) ?! \rightarrow y'' + a_1 y' + a_0 y = g(x)$
 ↳ (Particular solu.)

- if $g(x) = c \Rightarrow y_p(x) = \frac{c}{a_0}$
- if $g(x) = cx \Rightarrow y_p(x) = \frac{c}{a_0} \left[x - \frac{a_1}{a_0} \right]$
- if $g(x) = cx^2 \Rightarrow y_p(x) = \frac{c}{a_0} \left[x^2 - \frac{2a_1}{a_0} x + \frac{2(a_1^2 - a_0)}{a_0^2} \right]$

example : solve $y'' + 3y' + 2y = 1$
 $a_1 = 3, a_0 = 2, g(x) = 1$

$$\bullet r_1 = \frac{-3 \pm \sqrt{9-4(2)}}{2} \quad \bullet r_2 = \frac{-3 - \sqrt{9-4(2)}}{2}$$

$$r_1 = -1$$

$$r_2 = -2$$

$r_1 \neq r_2$
 ↳
 (real roots)

- $y_c(x) = Ae^{-x} + Be^{-2x}$
- $y_p(x) = 1/a_0 = 1/2$

$$\Rightarrow y(x) = y_c(x) + y_p(x) = Ae^{-x} + Be^{-2x} + \frac{1}{2}$$

- if $g(x) = bx \leadsto y_p(x) = b/2 \left[x - 3/2 \right]$
 $= 3 \left(x - 3/2 \right)$

example : solve $y'' + y' + y = 5x^2$
 $a_1 = 1, a_0 = 1, g(x) = 5x^2$

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = -1/2 \pm \frac{\sqrt{3}}{2} i$$

$$r_1 = -1/2 + \frac{\sqrt{3}}{2} i, r_2 = -1/2 - \frac{\sqrt{3}}{2} i$$

$$\alpha = -1/2, \beta = \sqrt{3}/2$$

$$\bullet y_c(x) = (A \cos \sqrt{3}/2 x + B \sin \sqrt{3}/2 x) e^{-\frac{1}{2}x}$$

$$\bullet y_p(x) = \frac{5}{1} \left[x^2 - 2 + \frac{2(1-1)}{1^2} \right]$$

$$= 5x^2 - 10$$

$$\bullet y(x) = (A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x) e^{-\frac{1}{2}x} + 5x^2 - 10$$

note :

* if $g(x) = g_1 + g_2 + \dots$, we should find $y_p(x)$ for every g_i .

method (2) $y'' = f(x, y')$

$$\Rightarrow y'' + p(x)y' = g(x) \leadsto y'' + 2xy' = x^2$$

assume $P = y' \rightarrow P' = dP/dx = y''$

example:

$$y'' + 2xy' = x$$

$$\frac{dP}{dx} + 2xP = x \quad * \text{first order ODEs solved by integrating factor}$$

$$\mu = e^{\int 2x dx} = e^{x^2}$$

$$y = e^{-x^2} \int e^{x^2} \cdot x dx \quad * u = x^2 \rightarrow du = 2x dx$$

$$y = e^{-u} \int e^u \cdot x \frac{du}{2x}$$

$$= \frac{1}{2} e^{-x^2} \int e^u du = \frac{1}{2} e^{-x^2} (e^{x^2} + C)$$

$$= \frac{1}{2} + \frac{1}{2} C e^{-x^2} = \frac{1}{2} + A e^{-x^2} \quad * (A = \frac{1}{2} C)$$

$$P = \frac{1}{2} + A e^{-x^2} = dy/dx$$

$$\int dy = \int \left(\frac{1}{2} + A e^{-x^2} \right) dx$$

$$y = \frac{1}{2} x + A \int e^{-x^2} dx + B$$

$\hookrightarrow$ will be solved later

method (3) :

$$y'' + p(x)y' + q(x)y = g(x)$$

$$y'' = f(x, y, y')$$

assume x does not exist $\rightarrow y'' = f(y, y')$

example :

$$1 + yy'' + y'^2 = 0$$

$$y'' + 2y' - y = 0$$

$\rightarrow$ solu :

$$y' = P, \quad y'' = P' = dP/dx = dP/dy \cdot dy/dx$$

$$= dP/dy \cdot y' = d/dy \cdot P$$

$$\rightarrow y'' = P' = P dP/dy$$

example: $1 + yy'' + y'^2 = 0$

$$\rightarrow 1 + y P \frac{dP}{dy} + P^2 = 0$$

$$y P \frac{dP}{dy} = -(1 + P^2)$$

$$P \frac{dP}{(1 + P^2)} = -\frac{dy}{y}$$

$$\frac{1}{2} \ln(1 + P^2) = -\ln(y) + C$$

$$C = \frac{1}{2} \ln(1 + P^2) + \ln(y) \quad *2$$

$$C_1 = \ln(1 + P^2) + \ln(y^2)$$

$$C_1 = \ln[(1 + P^2) \cdot y^2] \leftarrow \exp$$

$$(1 + P^2)y^2 = C_2 = \alpha^2 \text{ (constant)}$$

$$1 + P^2 = \frac{\alpha^2}{y^2} = \left(\frac{\alpha}{y}\right)^2$$

$$\rightarrow P^2 = \left(\frac{\alpha}{y}\right)^2 - 1 \rightarrow P = \sqrt{\left(\frac{\alpha}{y}\right)^2 - 1} = y' = \frac{dy}{dx}$$

$$\frac{dy}{\sqrt{\left(\frac{\alpha}{y}\right)^2 - 1}} = dx \rightarrow (x+b)^2 + y^2 = \alpha^2 \quad \#$$

method (4):

$$xy'' = f(y', y/x)$$

example:

$$xy'' + y'^2 - (y/x)^2 = 0$$

assume $r = y/x \rightarrow y = xr$

$$y' = xr' + r$$

$$y'' = xr'' + r' + r' = xr'' + 2r'$$

$$x[xr'' + 2r'] + [xr' + r]^2 - r^2 = 0$$

$$[x^2r'' + 2xr'] + [x^2r'^2 + 2xr'r + r^2] - r^2 = 0$$

$$x^2r'' + 2xr' + 2xr'r + x^2r'^2 = 0 \quad \dots \dots \dots \square$$

assume $x = e^t \rightarrow t = \ln x$

$$r' = \frac{dr}{dx} = \frac{dr}{dt} \cdot \frac{dt}{dx} = \frac{dr}{dt} \left(\frac{1}{x}\right)$$

$$r'' = \frac{d^2r}{dx^2} = \left[\frac{d^2r}{dt^2} - \frac{dr}{dt} \right] \left(\frac{1}{x}\right)^2$$

(take it as it is)

$$\rightarrow x^2 \left[\frac{d^2 r}{dt^2} - \frac{dr}{dt} \right] \left(\frac{1}{x^2} \right) + 2x \left(\frac{dr}{dt} \right) \left(\frac{1}{x} \right) +$$

$$2x r \left(\frac{dr}{dt} \right) \left(\frac{1}{x} \right) + x^2 \left(\frac{dr}{dt} \right)^2 \left(\frac{1}{x^2} \right) = 0$$

$$\rightarrow \frac{d^2 r}{dt^2} - \frac{dr}{dt} + 2 \frac{dr}{dt} + 2r \frac{dr}{dt} + \left(\frac{dr}{dt} \right)^2 = 0$$

(assume) $P = \frac{dr}{dt}$

$$\frac{dr^2}{dt^2} = \frac{dP}{dt} = \frac{dP}{dr} \cdot \frac{dr}{dt} = \frac{dP}{dr} \cdot P$$

$$\rightarrow P \frac{dP}{dr} - P + 2P + 2rP + P^2 = 0$$

$$P \frac{dP}{dr} + P + 2rP + P^2 = 0$$

$$P \frac{dP}{dr} + P(1+2r) + P^2 = 0 \quad (\div P)$$

$$\frac{dP}{dr} + P = -(1+2r) \approx \text{ODE solved by integrating factor}$$

توقف
الخطم

$$\Rightarrow \mu = e^{\int dr} = e^r$$

$$P = \frac{1}{\mu} \int \mu Q(r) dr$$

$$= e^{-r} \int e^r (1+2r) dr \rightarrow \text{by parts}$$

$$P = 1 - 2r + ce^{-r} \quad \#$$

$$\Rightarrow P = \frac{dr}{dt} = x \frac{dr}{dx} = x \frac{d(y/x)}{dx}$$

$$\frac{dy}{dx} - \frac{y}{x} = 1 - 2 \left(\frac{y}{x} \right) + c_2 e^{(-y/x)} \rightarrow \text{take it as it is}$$



* Solution of 2nd ODE's:

- series solution of ODE's - Frobenius method

$$P_2(x)y'' + P_1(x)y' + P_0(x)y = 0$$

$$\rightarrow \text{let } y = x^c \sum_{n=0}^{\infty} A_n x^{np} = x^c [A_0 x^0 + A_1 x^p + A_2 x^{2p} \dots]$$

assume $y = x^c \rightarrow y' = c x^{c-1} \rightarrow y'' = c(c-1) x^{c-2}$

$$\rightarrow P_2 c(c-1) x^{c-2} + P_1 c x^{c-1} + P_0 x^c = 0$$

$$\hookrightarrow f(c) x^r + g(c) x^{r+p} = 0$$

this term approaches zero faster than the other term. So we can neglect it.
that gives

$$f(c) x^r = 0, x^r \neq 0 \text{ so } f(c) = 0$$

$\hookrightarrow$ from the difference between x^r, x^{r+p} , we defined p

$\hookrightarrow$ and c can determine from $f(c) = 0 \rightarrow c_1, c_2$.

$$\rightarrow A_{k+1} = (-1)^{k+1} A_0 \frac{g(c_1)g(c_1+p)g(c_1+2p)\dots}{f(c_1+p)f(c_1+2p)f(c_1+3p)\dots}$$

Find A for c_1 and A for c_2

* these are three cases based on c values

case (1): c_1, c_2 are real and do not differ by integer

$\hookrightarrow c_1 - c_2$ should be fraction. ex. $c_1 = 0, c_2 = 1/2$

$c_1 = 3, c_2 = 3.4 / c_1 = 1, c_2 = 2 \rightarrow$ not valid in this case

Solu. $y = Ay_1 + By_2$

(A_{k+1}) from $c_1 \rightarrow$ (B_{k+1}) from c_2

$$y_1 = x^{c_1} + A_1 x^{c_1+p} + A_2 x^{c_1+2p} + \dots$$

$$y_2 = x^{c_2} + B_1 x^{c_2+p} + B_2 x^{c_2+2p} + \dots$$

example:

$$2xy'' + y' + y = 0 \leftrightarrow P_2 y'' + P_1 y' + P_0 y = 0$$

$$P_2 = 2x, P_1 = 1, P_0 = 1$$

$$y = x^c, y' = c x^{c-1}, y'' = c(c-1) x^{c-2}$$

$$\rightarrow 2xc(c-1) x^{c-2} + c x^{c-1} + x^c = 0$$

$$= 2c(c-1) x^{c-1} + c x^{c-1} + x^c = 0$$

$$[2c(c-1) + c] x^{c-1} + x^c = 0$$



$$f(c) x^{c-1} + g(c) x^c = 0$$

$$f(c) = 2c^2 - 2c + c = 0$$

$$2c^2 - c = 0$$

$$c(2c-1) = 0$$

$$c = 0, c = .5$$

$$g(c) = 1 \quad \leadsto \quad \text{note: } \therefore \text{ value of } g(c) \text{ always should be } (1)$$

$$p = c - (c-1) = 1$$

→ for first solu. $c_1 = 0$

$$y_1 = x^c + A_1 x^{c+p} + A_2 x^{c+2p} + \dots$$

$$= x^0 + A_1 x^{0+1} + A_2 x^{0+2(1)} \dots$$

$$= 1 + A_1 x + A_2 x^2 + \dots$$

$$\rightarrow k=0 \rightarrow A_1 = (-1)^1 A_0 \frac{g(c_1)}{f(c_1+p)}$$

we take first

$$\text{term only } (f(c_1+p))(g(c_1)) \mid = -A_0 \frac{1}{2[0+1]^2 - [0+1]}$$

$$= -A_0$$

$$f(c) = 2c^2 - c, \quad f(c_1+p) = 2[c_1+p]^2 - [c_1+p]$$

$$\rightarrow k=1 \rightarrow A_2 = (-1)^2 A_0 \frac{g(c_1) g(c_1+p)}{f(c_1+p) f(c_1+2p)}$$

$$= A_0 \frac{1 * 1}{1 * 6} = \frac{A_0}{6}$$

$$\rightarrow k=2 \rightarrow A_3 = (-1)^3 A_0 \frac{g(c_1) g(c_1+p) g(c_1+2p)}{f(c_1+p) f(c_1+2p) f(c_1+3p)}$$

$$= -A_0 \frac{1 * 1 * 1}{1 * 6 * 15} = -\frac{A_0}{90}$$

$$\rightarrow y_1 = A_0 x^c + A_1 x^{c_1+p} + A_2 x^{c_1+2p} + A_3 x^{c_1+3p}$$

$$= A_0 x^0 + (-A_0) x^1 + \frac{A_0}{6} x^2 + \frac{(-A_0)}{90} x^3$$

$$= A_0 - A_0 x + \frac{A_0}{6} x^2 - \frac{A_0}{90} x^3$$

$$y_1 = A_0 \left[1 - x + \frac{1}{6} x^2 - \frac{1}{90} x^3 \right]$$

$$\rightarrow B_{k+1} = (-1)^{k+1} B_0 \frac{g(c_2+p)g(c_2+2p)\dots}{f(c_2+p)f(c_2+2p)f(c_2+3p)\dots}$$

$$k=0 \rightarrow B_1 = (-1)^1 B_0 \frac{1}{(\frac{1}{2}+1)(2(\frac{3}{2})-1)}$$

$$= -B_0 \frac{1}{3} = -\frac{B_0}{3}$$

$$k=2 \rightarrow B_3 = (-1)^3 B_0 \frac{1}{(3)(10)(\frac{1}{2}+3)(2(\frac{7}{2})-1)}$$

$$= -\frac{B_0}{(3)(10)(21)} = -\frac{B_0}{630}$$

$$\rightarrow y_2 = B_0 x^{\frac{1}{2}} \left[1 - \frac{1}{3}x + \frac{1}{80}x^2 - \frac{1}{630}x^3 \right]$$

$$\rightarrow y = y_1 + y_2$$

Case (2): Both Roots are equal $c_1 = c_2 = c$

$$\text{Solu:- } y = \chi y_1(x; c) + \delta \left(\frac{\partial y_1}{\partial c} \right)_{c=c_1}$$

$$y_1 = x^r \sum a_n x^n$$

$$y_2 = y_1(x) \ln x + x^r \sum b_n x^n$$

$$\text{example:- } xy'' + y' + xy = 0 \quad P_2 = x, P_1 = 1, P_0 = x$$

$$y = x^c, y' = c x^{c-1}, y'' = c(c-1) x^{c-2}$$

$$\rightarrow xc(c-1)x^{c-2} + cx^{c-1} + x \cdot x^c = 0$$

$$(c^2 - c)x^{c-1} + cx^{c-1} + x^{c+1} = 0$$

$$c^2 x^{c-1} + (1)x^{c+1} = 0$$

$$f(c) = c^2 \quad g(c) = 1 \quad P = (c+1) - (c-1) = 2$$

$$\rightarrow c_1 = c_2 = 0 \quad \text{repeated roots}$$

$$\rightarrow y_1 = A_0 x^r + A_1 x^{r+p} + A_2 x^{r+2p} + A_3 x^{r+3p}$$

$$k=0 \rightarrow A_1 = (-1)^1 A_0 \frac{1}{(c+2)^2} = -A_0/4$$

$$k=1 \rightarrow A_2 = (-1)^2 A_0 \frac{(1)(1)}{(c+2)^2(c+4)^2} = A_0/64$$

$$k=2 \rightarrow A_3 = (-1)^3 A_0 \frac{(1)}{(c+2)^2(c+4)(c+6)^2} = -\frac{A_0}{(4)(16)(36)} = -\frac{A_0}{2304}$$

$$\begin{aligned} \rightarrow y_1 &= A_0 x^c + A_1 x^{c+2} + A_2 x^{c+4} \\ &= A_0 x^c - \frac{A_0}{(c+2)^2} x^{c+2} + \frac{A_0}{(c+2)^2(c+4)^2} x^{c+4} + \dots \\ &= A_0 x^c \left[1 - \frac{x^2}{(c+2)^2} + \frac{x^4}{(c+2)^2(c+4)^2} + \dots \right] \end{aligned}$$

$$y_1(c=0) = A_0 \left[1 - \frac{x^2}{4} + \frac{x^4}{64} + \dots \right]$$

$$\begin{aligned} \rightarrow y_2 &= \frac{dy_1}{dc} = \frac{d}{dc} \left[A_0 x^c \left[1 - \frac{x^2}{(c+2)^2} + \frac{x^4}{(c+2)^2(c+4)^2} \right] \right] \\ &= y_1 \ln x + x^c \left[1 + \frac{2x^2}{(c+2)^2} - \frac{2x^4}{(c+2)^2(c+4)^2} \right] \end{aligned}$$

$$\rightarrow P_2(x) y'' + P_1(x) y' + P_0(x) y = 0$$

$$y = x^c \rightarrow f(c) = x^r + g(c) x^{r+p} = 0$$

$f(c) = 0 \rightarrow$ roots $\rightarrow$ case 1 $c_1 \neq c_2$ (differ by fraction)

$\hookrightarrow$ case 2 $c_1 = c_2$

Case (3) :: $c_1 \neq c_2$ differ by integer

$$y = A y_1(x, c_2) + B \frac{d}{dc} \left[(c - c_1) y(x, c) \right]_{c=c_2}$$

* c_1 = Smallest root , * c_2 = largest root

$$\bullet A_{k+1} = (-1)^{k+1} A_0 \frac{g(c) g(c+p) g(c+2p)}{f(c+p) f(c+2p) f(c+3p)}$$

(Same for B)

example :: $xy'' + y = 0$

$$P_2 = x, P_1 = 0, P_0 = 1$$

$$\rightarrow y = A y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r} \rightarrow \text{largest root}$$

$$\rightarrow y = x^c, y' = c x^{c-1}, y'' = c(c-1) x^{c-2}$$

$$x c(c-1) x^{c-2} + x^c = 0$$

$$c(c-1) x^{c-1} + x^c = 0$$

$$\rightarrow f(c) = c(c-1), g(c) = 1, p=1$$

$$y = A_0 x^c + A_1 x^{c+p} + A_2 x^{c+2p}$$

$$\rightarrow k=0, A_1 = (-1)^1 A_0 \frac{1}{(c+1)(c+1-1)}$$

$$= \frac{-A_0}{c(c+1)}$$

note ::

$$f(c) = c(c-1)$$

$$f(c+p) = (c+p)(c+p-1)$$

$$f(c+2p) = (c+2p)(c+2p-1)$$

$$K=1, A_2 = \frac{(-1)^2 A_0}{[c(c+1)][(c+2)(c+1)]} = \frac{A_0}{c(c+1)^2(c+2)} = \text{neg}$$

$$y = A_0 x^c + A_1 x^{c+1} + A_2 x^{c+2} + \dots$$

$$= A_0 x^c - \frac{A_0}{c(c+1)} x^{c+1} + \frac{A_0}{c(c+1)^2(c+2)} x^{c+2} + \dots$$

$$f(c) = c(c-1) = 0 \rightarrow c_1 = 0 \text{ (smallest)}, c_2 = 1 \text{ (largest)}$$

$$\rightarrow y = A_0 x^c \left[1 - \frac{x}{c(c+1)} + \frac{x^2}{c(c+1)^2(c+2)} + \dots \right]$$

$$y_1 = A_0 x \left[1 - \frac{x}{1(2)} + \frac{x^2}{1(2)^2(3)} \right]$$

(use largest)

$$= A_0 x \left[1 - \frac{x}{2} + \frac{x^2}{12} \right]$$

$$y_2 = A_0 x^c - \frac{A_0 x^{c+1}}{c(c+1)} + \dots \Big|_{c=\text{largest}} + (c-c_1) \left[\frac{A_0 x^{c+1}}{c(c+1)^2} + \frac{A_0 x^{c+1}}{c^2(c+1)} \right]$$

Laplace Transformation

change t -domain $\rightarrow s$ -domain

$$f(t) \longrightarrow F(s)$$

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

Example (1)

$$f(t) = 1$$

$$L[1] = \int_0^{\infty} e^{-st} (1) dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}$$

Example (2)

$$L[k] = \int_0^{\infty} e^{-st} k dt = k \int_0^{\infty} e^{-st} dt = \frac{k}{s}$$

Example (3)

$$L[e^{\alpha t}] = \int_0^{\infty} e^{-st} e^{\alpha t} dt = \int_0^{\infty} e^{(\alpha-s)t} dt = \frac{1}{\alpha-s} e^{(\alpha-s)t} \Big|_0^{\infty} = \frac{1}{(s-\alpha)}$$

Example (4)

$$f(t) = kt \rightarrow F(s) = ?$$

$$\mathcal{L}[kt] = \int_0^{\infty} e^{-st} kt dt = k \int_0^{\infty} t e^{-st} dt$$

$$u = t \quad dv = e^{-st} dt$$

$$du = dt \quad v = -\frac{1}{s} e^{-st}$$

$$\begin{aligned} \int u dv &= uv \Big|_0^{\infty} - \int_0^{\infty} v du = \frac{-t}{s} e^{-st} \Big|_0^{\infty} + \int_0^{\infty} \frac{1}{s} e^{-st} dt \\ &= -\frac{1}{s^2} e^{-st} \Big|_0^{\infty} = \frac{1}{s^2} \end{aligned}$$

$$\mathcal{L}[kt] = \frac{k}{s^2}$$

Example (5)

$$f(t) = \sin \omega t \rightarrow F(s) = \frac{\omega}{s^2 + \omega^2} \quad \text{X}$$

$$\sin \omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

$$\mathcal{L}[f(t)] = \mathcal{L}[\sin \omega t] = \mathcal{L}\left[\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right] = \frac{1}{2i} \mathcal{L}[e^{i\omega t} - e^{-i\omega t}]$$

$$\mathcal{L}[\sin \omega t] = \frac{1}{2i} \int_0^{\infty} e^{-st} [e^{i\omega t} - e^{-i\omega t}] dt$$

$$= \frac{1}{2i} \left\{ \int_0^{\infty} e^{-st} e^{i\omega t} dt - \int_0^{\infty} e^{-st} e^{-i\omega t} dt \right\}$$

$$= \frac{1}{2i} \left\{ \int_0^{\infty} e^{-(s-i\omega)t} dt - \int_0^{\infty} e^{-(s+i\omega)t} dt \right\}$$

$$= \frac{1}{2i} \left\{ \frac{1}{s-i\omega} - \frac{1}{s+i\omega} \right\}$$

$$= \frac{1}{2i} \left\{ \frac{(s+i\omega) - (s-i\omega)}{s^2 + \omega^2} \right\}$$

$$= \frac{1}{2i} \left\{ \frac{2i\omega}{s^2 + \omega^2} \right\} = \frac{\omega}{s^2 + \omega^2}$$

$$\begin{aligned} (s-i\omega)(s+i\omega) &= s^2 - i^2 \omega^2 = s^2 - (-1) \omega^2 \\ &= s^2 + \omega^2 \end{aligned}$$



[HOMEWORK]

$$F(t) = te^{-\infty t}$$

$$F[te^{-\infty t}] = \int_0^{\infty} e^{-st} (te^{-\infty t}) dt$$

$$= \int_0^{\infty} te^{-(s+\infty)t} dt$$

$$\begin{array}{l} u = t \\ du = dt \end{array} \quad \begin{array}{l} dv = e^{-(s+\infty)t} \\ v = \frac{e^{-(s+\infty)t}}{-(s+\infty)} \end{array}$$

$$\int_0^{\infty} u dv = uv \Big|_0^{\infty} - \int_0^{\infty} v du$$

$$= \frac{-te^{-(s+\infty)t}}{s+\infty} \Big|_0^{\infty} + \int_0^{\infty} \frac{e^{-(s+\infty)t}}{(s+\infty)} dt$$

$$= 0 + \frac{-e^{-(s+\infty)t}}{(s+\infty)^2} \Big|_0^{\infty}$$

$$= \frac{1}{(s+\infty)^2}$$

$$\mathcal{L}\left[\frac{dx}{dt}\right] = \int_0^{\infty} \left(\frac{dx}{dt}\right) e^{-st} dt$$

let :

$$u = e^{-st}$$

$$du = -s e^{-st} dt$$

$$dv = \frac{dx}{dt}$$

$$v = x$$

$$\int_0^{\infty} \left(\frac{dx}{dt}\right) e^{-st} dt = x e^{-st} \Big|_0^{\infty} - \int_0^{\infty} -s x e^{-st} dt$$

$$= -x(t=\infty) + s \int_0^{\infty} x e^{-st} dt$$

$$= -x(t=\infty) + s X(s)$$

$$\mathcal{L}\left[\frac{dx}{dt}\right] = sX(s) - x(t=0)$$

note ::

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt = F(s)$$

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] = \int_0^{\infty} \left(\frac{d^2x}{dt^2}\right) e^{-st} dt$$

$$u = e^{-st}$$

$$dv = \frac{d^2x}{dt^2} dt$$

$$du = -s e^{-st} dt$$

$$v = \frac{dx}{dt} = x'(t)$$

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] = x'(t) e^{-st} \Big|_0^{\infty} - \int_0^{\infty} -s x'(t) e^{-st} dt$$

$$= -x'(t=\infty) + s \int_0^{\infty} x'(t) e^{-st} dt$$

$$= -x'(t=\infty) + s [sX(s) - x(t=0)]$$

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] = s^2 X(s) - s x(0) - x'(0)$$

Solve :: $\frac{d^2y}{dx^2} + 4y = 3$ * given that $y(0) = y'(0) = 1$

$$[s^2 y(s) - s y(0) - y'(0)] + 4y(s) = \frac{3}{s}$$

$$s^2 y(s) + 4y(s) - s - 1 = \frac{3}{s}$$

$$y(s) [s^2 + 4] = \frac{3}{s} + s + 1$$

$$y(s) = \frac{3/s + s + 1}{s^2 + 4} = \frac{3/s}{s^2 + 4} + \frac{s}{s^2 + 4} + \frac{1}{s^2 + 4}$$

$$= \frac{3}{s[s^2 + 4]} + \frac{s}{s^2 + 4} + \frac{1}{s^2 + 4}$$

$$y(t) = (\text{?}) + (\cos 2t) + \frac{1}{2} \sin 2t$$

$$\begin{array}{c} \frac{F(t)}{\frac{1}{\alpha} \sin \alpha t} \quad \frac{F(s)}{\frac{1}{\alpha} \frac{\alpha}{s^2 + \alpha^2}} \\ \frac{1}{s(s^2 + \alpha^2)} \end{array}$$

Solve: $dy/dt + y = 0 \sim dy/-y = dt = -\ln y = t, y = e^{-t}$

$$y'(0) = 1$$

$$s y(s) - y(0) + y(s) = 0$$

$$y(s) [s+1] = 1$$

$$y(s) = \frac{1}{s+1} \sim y(t) = e^{-t}$$

$$\begin{aligned} F(s) &= \frac{z(s)}{R(s)} = \frac{z(s)}{(s-p_1)^n (s-p_2)^{n-1} (s-p_3) (s-p_4)^m (s-p_5)^{m-1} \dots} \\ &= \frac{A_1}{(s-p_1)^n} + \frac{A_2}{(s-p_1)^{n-1}} + \frac{A_3}{(s-p_1)^{n-2}} \dots + \frac{B_1}{(s-p_2)^m} + \frac{B_2}{(s-p_2)^{m-1}} + \dots + \frac{C}{\dots} \end{aligned}$$

$$A_m = \lim_{s \rightarrow p_1} \left[\frac{d^{m-1}}{ds^{m-1}} (s-p_1)^n F(s) \right] \frac{1}{(m-1)!}$$

example:

$$\frac{3}{(x-2)(x+4)} = \frac{A}{(x-2)} + \frac{B}{(x+4)}$$

$$= \frac{A(x+4) + B(x-2)}{(x-2)(x+4)}$$

$$= \frac{Ax + 4A + Bx - 2B}{(x-2)(x+4)}$$

$$(A+B)x + (4A-2B) = 3 + 0x$$

$$A+B=0 \sim A=-B$$

$$4A-2B=3 \sim 4(-B)-2B=3$$

$$-4B-2B=3 \sim B = -\frac{1}{2}, A = \frac{1}{2}$$

$$\frac{3}{(x-2)(x+4)} = \frac{1/2}{(x-2)} + \frac{-1/2}{(x+4)}$$

example ::

$$F(s) = \frac{1}{(s-a)(s-b)} = \frac{A_1}{s-a} + \frac{B_1}{(s-b)}$$

$$A_1 = \lim_{s \rightarrow a} (s-a) \left[\frac{1}{(s-a)(s-b)} \right] = \frac{1}{a-b}$$

$$B_1 = \lim_{s \rightarrow b} (s-b) \left[\frac{1}{(s-a)(s-b)} \right] = \frac{1}{b-a}$$

$$F(s) = \frac{1/(a-b)}{s-a} + \frac{1/(b-a)}{s-b}$$

$$b = -4$$

$$a = 2$$

example ::

$$F(s) = \frac{1}{(s-a)^2(s-b)} = \frac{A_1}{(s-a)^2} + \frac{A_2}{(s-a)} + \frac{B_1}{(s-b)}$$

$$A_1 = \lim_{s \rightarrow a} \left[(s-a)^2 \frac{1}{(s-a)^2(s-b)} \right] = \frac{1}{(a-b)}$$

$$A_2 = \lim_{s \rightarrow a} \left[\frac{d}{ds} \left((s-a)^2 \frac{1}{(s-a)^2(s-b)} \right) \right] = \lim_{s \rightarrow a} \left[\frac{d}{ds} \left(\frac{1}{s-b} \right) \right]$$

$$= \frac{-(1)(1)}{(s-b)^2} \bigg|_{s \rightarrow a} = \frac{-1}{(a-b)^2}$$

$$B_1 = \lim_{s \rightarrow b} \left[(s-b) \frac{1}{(s-a)^2(s-b)} \right] = \frac{1}{(b-a)^2}$$

$$F(s) = \frac{1/(a-b)}{(s-a)^2} + \frac{-1/(a-b)^2}{(s-a)} + \frac{1/(b-a)^2}{s-b}$$

$$F(s) = \left(\frac{1}{a-b} \right) t e^{at} - \frac{1}{(a-b)^2} e^{at} + \frac{1}{(b-a)^2} e^{bt}$$

F(t)

k

t

t^{n-1}

$(n-1)!$

$\frac{1}{\sqrt{\pi t}}$

$2\sqrt{\frac{t}{\pi}}$

$\frac{1}{\alpha} \sin \alpha t$

$\cos \alpha t$

$\frac{1}{\alpha} \sinh(\alpha t)$

F(s)

k/s

1/s²

1/sⁿ, n=1,2,...

1/√s

s^{-3/2}

1/s²+κ²

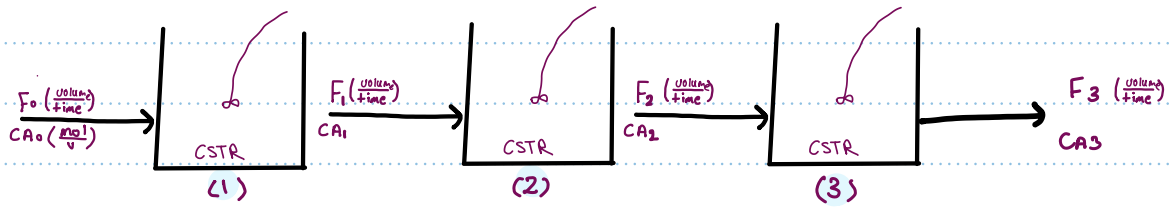
s/s²+κ²

1/s²-α²

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt = \lim_{b \rightarrow \infty} \int_0^b e^{-st} f(t) dt$$

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}$	2. e^{at}	$\frac{1}{s-a}$
3. $t^n, n=1,2,3,\dots$	$\frac{n!}{s^{n+1}}$	4. $t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	6. $t^{n-1/2}, n=1,2,3,\dots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \sqrt{\pi}}{2^n s^{n+1/2}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$	8. $\cos(at)$	$\frac{s}{s^2+a^2}$
9. $t \sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10. $t \cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11. $\sin(at) - at \cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12. $\sin(at) + at \cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13. $\cos(at) - at \sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14. $\cos(at) + at \sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15. $\sin(at+b)$	$\frac{s \sin(b) + a \cos(b)}{s^2+a^2}$	16. $\cos(at+b)$	$\frac{s \cos(b) - a \sin(b)}{s^2+a^2}$
17. $\sinh(at)$	$\frac{a}{s^2-a^2}$	18. $\cosh(at)$	$\frac{s}{s^2-a^2}$
19. $e^{at} \sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20. $e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21. $e^{at} \sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22. $e^{at} \cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23. $t^n e^{at}, n=1,2,3,\dots$	$\frac{n!}{(s-a)^{n+1}}$	24. $f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$
25. $u_c(t) = u(t-c)$ Heaviside Function	$\frac{e^{-cs}}{s}$	26. $\delta(t-c)$ Dirac Delta Function	e^{-cs}
27. $u_c(t) f(t-c)$	$e^{-cs} F(s)$	28. $u_c(t) g(t)$	$e^{-cs} \mathcal{L}\{g(t+c)\}$
29. $e^{at} f(t)$	$F(s-c)$	30. $t^n f(t), n=1,2,3,\dots$	$(-1)^n F^{(n)}(s)$
31. $\frac{1}{t} f(t)$	$\int_s^{\infty} F(u) du$	32. $\int_0^t f(v) dv$	$\frac{F(s)}{s}$
33. $\int_0^t f(t-\tau) g(\tau) d\tau$	$F(s) G(s)$	34. $f(t+T) = f(t)$	$\frac{\int_0^T e^{-st} f(t) dt}{1-e^{-sT}}$
35. $f'(t)$	$sF(s) - f(0)$	36. $f''(t)$	$s^2 F(s) - sf'(0) - f''(0)$
37. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$		



$$F_0 = F_1 = F_2 = F_3 \quad \left| \quad C_{A1}(0) = C_{A2}(0) = C_{A3}(0) = 0 \right.$$

$$U_1 = U_2 = U_3$$

Reactor (1) :

mol in - mol out + cons. = acc

$$C_A = \frac{N_A}{U} \rightarrow N_A = U C_A$$

$$\frac{U A_0}{\text{time}} - \frac{U A}{\text{time}} - r A U = \frac{d N_A}{dt}$$

$$F(C_{A0} - C_A) - R C_A U = U \frac{d C_A}{dt}$$

$$\frac{\text{Volume} \left(\frac{\text{mol}}{\text{Volume}} - \frac{\text{mol}}{\text{Volume}} \right)}{\text{time}}$$

$$U \frac{d C_A}{dt} = F C_{A0} - [F C_A + R C_A U]$$

$$= F C_{A0} - [F + R U] C_A$$

$$\frac{d C_{A1}}{dt} + \frac{[F + R U]}{U} C_{A1} = \frac{F C_{A0}}{U}$$

$$\frac{dy}{dx} + \alpha_1 y = \beta_1(1)$$

$$F(C_{A1} - C_{A2}) - R C_{A2} U$$

$$\frac{d C_{A2}}{dt} + \frac{[F + R U]}{U} C_{A2} = \frac{F C_{A1}}{U}$$

$$\frac{dz}{dx} + \alpha_2 z = \beta_2 y$$

$$\frac{d C_{A3}}{dt} + \frac{[F + R U]}{U} C_{A3} = \frac{F C_{A2}}{U}$$

$$\frac{dw}{dt} + \alpha_3 w = \beta_3 z$$

$$dy/dt + \alpha_1 y = \beta_1$$

$$dz/dt + \alpha_2 z = \beta_2 y$$

$$dw/dt + \alpha_3 w = \beta_3 z$$

$$\frac{dCA_1}{dt} + \frac{[F+RV]}{V} CA_1 = \frac{F CA_0}{V}$$

Laplace Transformation ::

$$\frac{dCA_1}{dt} + \alpha_1 CA_1 = \beta_1$$

$$[sCA_1(s) - CA_1(0)] + \alpha_1 CA_1(s) = \frac{\beta_1}{s}$$

$$sCA_1(s) + \alpha_1 CA_1(s) = \frac{\beta_1}{s}$$

$$CA_1 [s + \alpha_1] = \frac{\beta_1}{s}$$

$$CA_1 = \frac{\beta_1}{s(s + \alpha_1)} = \frac{A_1}{s} + \frac{B_1}{s + \alpha_1}$$

$$A_1(s + \alpha_1) + B_1(s) = \beta_1 + 0s$$

$$A_1\alpha_1 + A_1s + B_1(s) = \beta_1 + 0s$$

$$A_1\alpha_1 = \beta_1 \quad A_1 = \beta_1/\alpha_1$$

$$(A_1 + B_1)s = 0s \quad B_1 = -A_1 = -\beta_1/\alpha_1$$

$$CA_1 = \frac{\beta_1/\alpha_1}{s} + \frac{-\beta_1/\alpha_1}{s + \alpha_1}$$

$$CA_1(s) = \frac{\beta_1}{\alpha_1} \left[\frac{1}{s} - \frac{1}{s + \alpha_1} \right]$$

$$CA_1(t) = \frac{\beta_1}{\alpha_1} [1 - e^{-\alpha_1 t}] \rightsquigarrow \text{table}$$



$$1) dCA_1/dt + \alpha_1 CA_1 = \beta_1$$

$$2) dCA_2/dt + \alpha_2 CA_2 = \beta_2 CA_1$$

$$3) dCA_3/dt + \alpha_3 CA_3 = \beta_3 CA_2$$

$$1) CA_1(s) = \frac{\beta_1}{s(s+\alpha_1)} = \frac{\beta_1}{\alpha_1} \left[\frac{1}{s} - \frac{1}{s+\alpha_1} \right] \rightarrow CA_1(t) = \frac{\beta_1}{\alpha_1} [1 - e^{-\alpha_1 t}]$$

$$2) [s CA_2(s) - CA_2(0)] + \alpha_2 CA_2(s) = \frac{\beta_2 \beta_1}{s(s+\alpha_1)} \rightarrow (s+\alpha_2) CA_2(s) = \frac{\beta_1 \beta_2}{s(s+\alpha_1)} \rightarrow$$

$$\rightarrow CA_2(s) = \frac{\beta_1 \beta_2}{s(s+\alpha_1)(s+\alpha_2)} \quad \text{then} \quad CA_2(s) = \frac{A_1}{s} + \frac{B_1}{s+\alpha_1} + \frac{C_1}{s+\alpha_2} = \frac{\beta_1 \beta_2}{s(s+\alpha_1)(s+\alpha_2)}$$

$$A_1 = \lim_{s \rightarrow 0} s \frac{\beta_1 \beta_2}{s(s+\alpha_1)(s+\alpha_2)} = \frac{\beta_1 \beta_2}{\alpha_1 \alpha_2}$$

$$B_1 = \lim_{s \rightarrow -\alpha_1} (s+\alpha_1) \frac{\beta_1 \beta_2}{s(s+\alpha_1)(s+\alpha_2)} = \frac{-\beta_1 \beta_2}{(\alpha_2 - \alpha_1) \alpha_1}$$

$$C_1 = \lim_{s \rightarrow -\alpha_2} (s+\alpha_2) \frac{\beta_1 \beta_2}{s(s+\alpha_1)(s+\alpha_2)} = \frac{-\beta_1 \beta_2}{(\alpha_1 - \alpha_2) \alpha_2}$$

$$CA_2(s) = \beta_1 \beta_2 \left[\frac{1/\alpha_1 \alpha_2}{s} - \frac{1}{\alpha_1 (\alpha_2 - \alpha_1) (s+\alpha_1)} - \frac{1}{(\alpha_1 - \alpha_2) \alpha_2 (s+\alpha_2)} \right]$$

$$CA_2(t) = \beta_1 \beta_2 \left[\frac{1}{\alpha_1 \alpha_2} - \frac{1}{\alpha_1 (\alpha_2 - \alpha_1)} e^{-\alpha_1 t} - \frac{1}{(\alpha_1 - \alpha_2) \alpha_2} e^{-\alpha_2 t} \right]$$

$$3) [s CA_3(s) - CA_3(0)] + \alpha_3 CA_3(s) = \frac{\beta_3 \beta_2 \beta_1}{s(s+\alpha_1)(s+\alpha_2)}$$

$$CA_3(s) [s+\alpha_3] = \dots$$

$$CA_3(s) = \frac{\beta_1 \beta_2 \beta_3}{s(s+\alpha_1)(s+\alpha_2)(s+\alpha_3)} = \frac{A_1}{s} + \frac{B_1}{s+\alpha_1} + \frac{C_1}{s+\alpha_2} + \frac{D_1}{s+\alpha_3}$$

$$A_1 = \frac{\beta_1 \beta_2 \beta_3}{\alpha_1 \alpha_2 \alpha_3}, \quad B_1 = \frac{-\beta_1 \beta_2 \beta_3}{\alpha_1 (\alpha_2 - \alpha_1) (\alpha_3 - \alpha_1)}, \quad C_1 = \frac{-\beta_1 \beta_2 \beta_3}{\alpha_2 (\alpha_3 - \alpha_2) (\alpha_1 - \alpha_2)}$$

$$D_1 = \frac{-\beta_1 \beta_2 \beta_3}{\alpha_3 (\alpha_1 - \alpha_3) (\alpha_2 - \alpha_3)}$$

$$CA_3(t) = A_1 + B_1 e^{-\alpha_1 t} + C_1 e^{-\alpha_2 t} + D_1 e^{-\alpha_3 t}$$

$$= \frac{\beta_1 \beta_2 \beta_3}{\alpha_1 \alpha_2 \alpha_3} \left[1 - \frac{\alpha_2 \alpha_3}{(\alpha_2 - \alpha_1) (\alpha_3 - \alpha_1)} e^{-\alpha_1 t} - \frac{\alpha_1 \alpha_3}{(\alpha_3 - \alpha_2) (\alpha_1 - \alpha_2)} e^{-\alpha_2 t} - \frac{\alpha_1 \alpha_2}{(\alpha_1 - \alpha_3) (\alpha_2 - \alpha_3)} e^{-\alpha_3 t} \right]$$

example:

$$dy_1/dt - 2y_1 = 1 \rightarrow y_1(0) = 0, y_2(0) = 1$$

$$dy_2/dt + y_2 = y_1 \quad sy_1(s) - y_1(0) - 2y_1(s) = \frac{1}{s}$$

$$y_1(s) = \frac{1}{s(s-2)} = \frac{A_1}{s} + \frac{B_1}{s-2}$$

$$y_1(t) = -\frac{1}{2} [1 - e^{2t}]$$

$$[sy_2(s) - y_2(0)] + y_2(s) = y_1(s)$$

$$[s+1]y_2(s) = \frac{1}{s(s-2)} + \frac{1}{1}$$

$$= \frac{1 + s(s-2)}{s(s-2)}$$

$$= \frac{s^2 - 2s + 1}{s(s-2)}$$

$$y_2(s) = \frac{s^2 - 2s + 1}{s(s+1)(s-2)} = \frac{A_1}{s} + \frac{B_1}{s+1} + \frac{C_1}{s-2}$$

$$A_1 = \lim_{s \rightarrow 0} s \frac{s^2 - 2s + 1}{s(s+1)(s-2)} = -\frac{1}{2}$$

$$C_1 = \lim_{s \rightarrow 2} (s-2) \frac{s^2 - 2s + 1}{s(s+1)(s-2)} = \frac{1}{6}$$

$$B_1 = \lim_{s \rightarrow -1} (s+1) \frac{s^2 - 2s + 1}{s(s+1)(s-2)} = \frac{4}{3}$$

$$A_1 = -\frac{1}{2}, B_1 = \frac{1}{2}$$

$$y_2(s) = -\frac{1}{2} \left(\frac{1}{s}\right) + \frac{4}{3} \frac{1}{s+1} + \frac{1}{6} \left(\frac{1}{s-2}\right)$$

$$y_2(t) = -\frac{1}{2} + \frac{4}{3} e^{-t} + \frac{1}{6} e^{2t}$$

example::

$$dy_1/dt + 4y_1 = y_2$$

$$dy_2/dt + 10y_2 = y_1$$

$$\rightarrow y_1(0) = y_2(0) = 0$$

$$[sy_1(s) + y_1(0)] + 4y_1(s) = y_2(s)$$

$$[sy_2(s) + y_2(0)] + 10y_2(s) = y_1(s)$$

$$y_1[s+4] = y_2$$

$$y_2[s+10] = y_1$$

$$y_1[s+4][s+10] = y_1$$

$$y_1[] [] - y_1 = 0$$

$$y_1[(s+4)(s+10)-1] = 0$$

$$\left. \begin{array}{l} y_1(s) = 0 \rightarrow y_1(t) = 0 \\ y_2[s+10] = 0 \rightarrow y_2(t) = 0 \end{array} \right\} \text{trivial solution}$$

$$y_2 \left[\frac{(s+10)(s+4)-1}{s+4} \right] = \frac{1}{s+4}$$

$$\rightarrow y_1(0) = 1, y_2(0) = 0$$

$$y_2 = \frac{(s+4)^2}{(s+10)(s+4)-1} = \frac{1}{(s+3.8)(s+10.15)} = \frac{A}{s+3.8} + \frac{B}{s+10.15} = \frac{1/6.35}{s+3.8} + \frac{-1/6.35}{s+10.15}$$

$$A(s+10.15) + B(s+3.8) = 1$$

$$(A+B)s + 10.15A + 3.8B = 1 + 0s$$

$$A = -B$$

$$10.15A + 3.8B = 1$$

$$-10.15B + 3.8B = 1$$

$$-6.35B = 1 \rightarrow B = \frac{-1}{6.35} \rightarrow A = \frac{1}{6.35}$$

$$(s+10)(s+4)-1 = s^2 + 14s + 39$$

$$s = \frac{-14 \pm \sqrt{196 - 4(1)(39)}}{2}$$

$$= \frac{-14 \pm 6.3}{2}$$

$$s_1 = -3.8, s_2 = -10.15$$

$$y_2(s) = \frac{1}{6.35} \left[\frac{1}{s+3.8} - \frac{1}{s+10.15} \right]$$

$$y_2(t) = \frac{1}{6.35} [e^{-3.8t} - e^{-10.15t}]$$

$$sy_1(s) - y_1(0) + 4y_1(s) = y_2(s)$$

$$y_1(s)[s+4]-1 = \frac{1}{(s+3.8)(s+10.15)}$$

$$y_1(s)[s+4] = \frac{(s+10)(s+4)}{(s+3.8)(s+10.15)}$$

$$y_1(s) = \frac{s+10}{(s+3.8)(s+10.15)}$$

$$= \frac{A}{s+3.8} + \frac{B}{s+10.15}$$



example:

$$\frac{d^3 y}{dt^3} \quad \mathcal{L} \left[\frac{d^3 y}{dt^3} \right] = \int_0^\infty \frac{d^3 y}{dt^3}$$

$$u = e^{-st}$$

$$du = -s e^{-st}$$

$$dv = \frac{d^3 y}{dt^3}$$

$$dt \quad v = \frac{d^2 y}{dt^2}$$

$$e^{-st} \frac{d^2 y(t)}{dt^2} \Big|_0^\infty - \int_0^\infty \left[\frac{d^2 y}{dt^2} \right] (-s e^{-st})$$

$$y''(0) + s \int_0^\infty \left(\frac{d^2 y}{dt^2} \right) e^{-st}$$

$$y''(0) + s [s^2 y(s) - s y(0) - y'(0)]$$

$$s^3 y(s) - s^2 y(0) - s y'(0) + y''(0)$$

$$\mathcal{L}[y'] = s y(s) - y(0)$$

$$\mathcal{L}[y''] = s^2 y(s) - s y(0) - y'(0)$$

$$\mathcal{L}[y'''] = s^3 y(s) - s^2 y(0) - s y'(0) - y''(0)$$

$$\mathcal{L}[y^{(4)}] = s^4 y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)$$

$$\mathcal{L}[y^{(5)}] = s^5 y(s) - s^4 y(0) - s^3 y'(0) - s^2 y''(0) - s y'''(0) - y^{(4)}(0)$$

$$\frac{d^3 y}{dt^3} + y = 1 \quad y(0) = y'(0) = y''(0) = 1$$

$$s^3 y(s) - s^2 - s - 1 + y(s) = \frac{1}{s}$$

$$y(s) [s^3 + 1] = \frac{1}{s} + \frac{s^2}{1} + \frac{s}{1} + \frac{1}{1}$$

$$= \frac{1 + s^3 + s^2 + s}{s}$$

$$y(s) [s^3 + 1] = \frac{s^3 + s^2 + s + 1}{s}$$

$$y(s) = \frac{s^3 + s^2 + s + 1}{s(s^3 + 1)}$$

4 terms will give me
continue the solution



Integral functions:

1) error functions

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\beta^2} d\beta$$

$$\operatorname{erf}(x) = 1 - (\alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3) e^{-x^2}$$

$$t = \frac{1}{1 + p x}$$

$$p = 0.47047$$

$$\alpha_1 = 0.34802$$

$$\alpha_2 = -0.09587$$

$$\alpha_3 = 0.74785$$

example:

$$\int_0^5 e^{-x^2} dx \leftrightarrow \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$$

$$t = \frac{1}{1 + 47(5)} = 0.298$$

$$\int_0^5 e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \operatorname{erf}(x)$$

$$= \frac{\sqrt{\pi}}{2} \left[1 - \{ 0.348(0.298) - 0.095(0.298)^2 + 0.747(0.298)^3 e^{-5^2} \} \right]$$

$$= \frac{\sqrt{\pi}}{2} \left[1 - \{ 0.103 - 8.4 \times 10^{-3} + 0.0198 \{ 1.38 \times 10^{-11} \} \} \right]$$

$$= \frac{\sqrt{\pi}}{2} [1 - 1.58 \times 10^{-12}] = \frac{\sqrt{\pi}}{2}$$

$\operatorname{erf}(0)$

= ?

Home
work

$$\operatorname{erf}(\infty) = 1$$

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{x^2} dx$$

$$\frac{d}{dx} \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

$$\int \operatorname{erf}(x) dx = x \operatorname{erf}(x) - \frac{1}{\sqrt{\pi}} e^{-x^2}$$

$$\operatorname{erf}(x) + \operatorname{erfc}(x) = 1 \rightarrow \operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\operatorname{erf}(0) = \frac{2}{\sqrt{\pi}} \int_0^0 e^{-t^2} dt = 0$$

$$\rightarrow \boxed{\operatorname{erf}(0) = 0}$$



Integral Functions

2) Gamma Functions.

$$\Gamma(x) = \frac{1}{x} \prod_{n=1}^{\infty} \frac{(1 + 1/n)^x}{(1 + x/n)} = \frac{1}{x} \left[\left\{ \frac{(1 + 1/1)^x}{(1 + x/1)} \right\} \left\{ \frac{(1 + 1/2)^x}{(1 + x/2)} \right\} \left\{ \frac{(1 + 1/3)^x}{(1 + x/3)} \right\} \dots \right]$$
$$= \int_0^x t^{x-1} e^{-t} dt$$

$$\Gamma(x+1) = x \Gamma(x)$$

For $x = \text{integer}$

$$\begin{aligned}\Gamma(n) &= (n-1) \Gamma(n-1) \\ &= (n-1)(n-2) \Gamma(n-2) \\ &= (n-1)(n-2)(n-3) \Gamma(n-3) \dots \Gamma(1) \\ &= (n-1)!\end{aligned}$$

example:

$$\Gamma(2) = (2-1)! = 1$$

$$\Gamma(5) = 4 \times 3 \times 2 \times 1$$

Integral Functions

3) Beta Function

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

$$= \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$$

example:

$$B(3, 7) = \frac{\Gamma(3) \Gamma(7)}{\Gamma(3+7)} = \frac{(2 \times 1)(6 \times 5 \times 4 \times 3 \times 2 \times 1)}{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{1}{9 \times 4 \times 7}$$

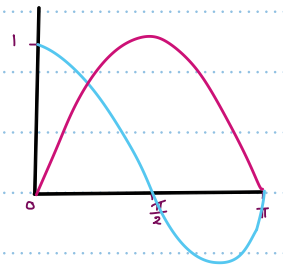


Fourier Series ::

any function, $f(x)$ can be written in the form of Fourier series

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad -\pi \leq x \leq \pi$$

$$f(x) = \frac{1}{2} a_0 + [a_1 \cos x + b_1 \sin x] \\ + [a_2 \cos 2x + b_2 \sin 2x] \\ + \dots$$

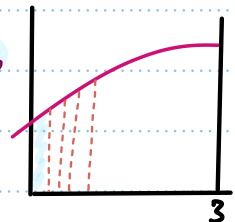


→ multiply by $\cos mx$::

$$f(x) \cos mx = \frac{1}{2} a_0 \cos mx + \sum_{n=1}^{\infty} a_n \cos nx \cos mx + \sum_{n=1}^{\infty} b_n \sin nx \cos mx$$

note :: (take it as it is)

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$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = 0 \quad m \neq n$$

$$\int_{-\pi}^{\pi} \cos nx \cos nx dx = \int_{-\pi}^{\pi} \cos^2 nx dx \quad \text{when } m = n$$

$$\int_{-\pi}^{\pi} \cos nx \sin mx dx = 0 \quad \text{any value of } n \text{ \& } m$$

$$\int_{-\pi}^{\pi} (\cos mx) f(x) dx = \frac{1}{2} a_0 \cos mx + \int_{-\pi}^{\pi} a_n \cos nx \cos mx dx + \int_{-\pi}^{\pi} b_n \sin nx \cos mx dx$$

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = \int_{-\pi}^{\pi} a_0 \cos mx dx + a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx$$

For $m \neq n \Rightarrow \int_{-\pi}^{\pi} f(x) \cos mx dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cos mx dx \rightarrow \text{Trivial solution}$

for $m = n \Rightarrow \int_{-\pi}^{\pi} f(x) \cos mx dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cos mx dx + a_n \int_{-\pi}^{\pi} \cos^2 mx dx$

Should be $\cos nx$?

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$



$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$



$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

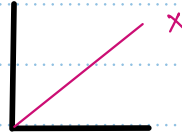




$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad -\pi \leq x \leq \pi$$

example:

$$f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ x & 0 \leq x \leq \pi \end{cases}$$



$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 f(x) dx + \frac{1}{\pi} \int_0^{\pi} f(x) dx \\ &= \frac{1}{\pi} \int_{-\pi}^0 0 dx + \frac{1}{\pi} \int_0^{\pi} x dx = \frac{\pi}{2} \end{aligned}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^0 0 \cos nx dx + \frac{1}{\pi} \int_0^{\pi} x \cos nx dx = \frac{1}{\pi n^2} [\cos n\pi - 1]_0^{\pi} \rightarrow (\text{يعني } \pi)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^0 0 \sin nx dx + \int_0^{\pi} x \sin nx dx = \left. -\frac{\cos nx}{n} \right|_0^{\pi}$$

note:

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad -\pi \leq x \leq \pi$$

$$\begin{aligned} &= \frac{1}{2} a_0 + a_1 \cos x + b_1 \sin x \\ &\quad + a_2 \cos 2x + b_2 \sin 2x \\ &\quad + a_3 \cos 3x + b_3 \sin 3x \\ &\quad + \dots \end{aligned}$$

Continue the solution of the example:

$$= \frac{1}{2} \left(\frac{\pi}{2} \right) + \frac{1}{\pi n^2} [\cos x - 1]_0^{\pi} \cos x + \left[\frac{-\cos x}{1} \right]_0^{\pi} \sin x$$

$$+ \frac{1}{\pi (2)^2} [\cos 2x - 1]_0^{\pi} \cos x + \left[\frac{-\cos 2x}{2} \right]_0^{\pi} \sin x$$

+ ...

$$= \frac{\pi}{4} + \frac{-2}{\pi} \cos x + 2 \sin x$$

+ ...

→ Continue the solution...

example: (homework)

$$f(x) = x^2$$

(not correct)

$f(x)$ is an even function

$$\text{Since } f(-x) = (-x)^2 = x^2$$

$$\therefore b_n = 0$$

$\Rightarrow$ we only need to calculate a_0 and a_n

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx$$

$$a_0 = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi} \cdot \frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \left[\frac{x^2 \sin(nx)}{n} + \frac{2x \cos(nx)}{n^2} - \frac{2 \sin(nx)}{n^3} \right]_0^{\pi}$$

$$a_n = \frac{2}{\pi} \left(\frac{2\pi(-1)^n}{n^2} \right) = \frac{4(-1)^n}{n^2}$$

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

$$x^2 = \frac{\pi^2}{3} - 4 \left(\cos(x) - \frac{\cos(2x)}{4} + \frac{\cos(3x)}{9} - \dots \right)$$

if we put $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

note:

* الحد غير صحيح



odd function $\rightarrow x, x^3, \sin x, \tan x$

$$f(-x) = -f(x)$$

$$f(x) = x, f(-x) = -x$$

even function $\rightarrow x^2, x^4, \cos x$

$$f(-x) = f(x)$$

$$f(x) = x^2 \rightarrow (-1)^2 = (1)^2 = 1$$

* ODD * ODD = even

$$x * x^3 = x^4$$

* ODD * even = ODD

$$x * x^2 = x^3$$

* even * even = even

$$x^2 * x^4 = x^6$$

$$\int_{-L}^L f(x) dx = \begin{cases} 0, & f(x) = \text{odd} \\ 2 \int_0^L f(x) dx, & f(x) = \text{even} \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

example :

$$f(x) = x + x^2$$

Find Fourier series for this function

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} [x + x^2] dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x dx + \int_{-\pi}^{\pi} x^2 dx \right] = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^3}{3} \right] = \frac{2}{3} \pi^2$$

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \cos nx \, dx + \int_{-\pi}^{\pi} x^2 \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^{\pi}$$

$$= \frac{4(-1)^n}{n^2}$$

note::

$$\cos nx \rightarrow n=0 \rightarrow \cos 0 = 1$$

$$n=1 \rightarrow \cos \pi = -1$$

$$n=2 \rightarrow \cos 2\pi = 1$$

$$n=3 \rightarrow \cos 3\pi = -1$$

$$\cos n\pi = (-1)^n$$

$$\int_{-\pi}^{\pi} x^2 \cos nx \, dx = 2 \int_0^{\pi} x^2 \cos nx \, dx$$

$$u = x^2 \quad du = 2x \, dx$$

$$v = \frac{1}{n} \sin nx$$

$$\frac{x^2}{n} \sin nx \Big|_0^{\pi} - \int_0^{\pi} \frac{2x}{n} \sin nx \, dx$$

$$\int_0^{\pi} x \sin nx \, dx \quad u = x \quad du = \sin nx \, dx$$

$$= -\frac{x}{n} \cos(nx) \Big|_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} \, dx$$

$$\int_0^{\pi} x^2 \cos nx \, dx = \frac{x^2}{n} \sin nx \Big|_0^{\pi} - \frac{2}{n} \left\{ -\frac{x}{n} \cos(nx) \Big|_0^{\pi} + \frac{1}{n^2} \sin(nx) \Big|_0^{\pi} \right\}$$

$$= \frac{\pi^2}{n} \sin(n\pi) - \frac{2}{n} \left\{ -\frac{\pi}{n} \cos n\pi \right\}$$

$$= \frac{2\pi}{n^2} \cos(n\pi)$$

example::

Convert $f(x) = e^x$

$$F(x) = \frac{a_0}{2} + \sum_1^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx$$

Solution:

$$\int e^{ax} \cos(bx) dx = e^{ax} \frac{a \cos(bx) + b \sin(bx)}{a^2 + b^2}$$

$$a_0 = \frac{e^{\pi} - e^{-\pi}}{\pi}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{e^x [\cos(nx) + n \sin(nx)]}{1+n^2} dx \\ &= \frac{1}{\pi} \left[\frac{e^{\pi} [(-1)^n \cos(n\pi) + n \sin(n\pi)]}{1+n^2} - \frac{e^{-\pi} [(-1)^n \cos(-n\pi) + n \sin(-n\pi)]}{1+n^2} \right] \\ &= \frac{e^{\pi} (-1)^n - e^{-\pi} (-1)^n}{\pi (1+n^2)} = \frac{(-1)^n [e^{\pi} - e^{-\pi}]}{\pi (1+n^2)} \end{aligned}$$

$$\int e^{ax} \sin(bx) dx = e^{ax} \frac{a \sin(bx) - b \cos(bx)}{a^2 + b^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin(nx) dx$$

$$\begin{aligned} &= \frac{1}{\pi} \left\{ \frac{e^x [\sin(nx) - n \cos(nx)]}{1+n^2} \right\}_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[\frac{e^{\pi} [\sin(n\pi) - n \cos(n\pi)]}{1+n^2} - \frac{e^{-\pi} [\sin(-n\pi) - n \cos(-n\pi)]}{1+n^2} \right] \end{aligned}$$

$$= \frac{1}{\pi (1+n^2)} [-ne^{\pi} (-1)^n + ne^{-\pi} (-1)^n]$$

$$b_n = \frac{n (-1)^n}{\pi (1+n^2)} [e^{-\pi} - e^{\pi}]$$

مطابقاً
الجدول
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convert $f(x) = \sin x$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin x dx = -\frac{1}{\pi} \cos x \Big|_{-\pi}^{\pi} = -\frac{1}{\pi} [-1 - (-1)] = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \cos(nx) dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \sin(nx) dx$$



$e^x \ln x$ → can't be solved

3^x

$$\frac{e^x - e^{-x}}{2}$$

Final

will be solved easily

only bn will be calculated

since it's an odd function

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partial Diff equ. :

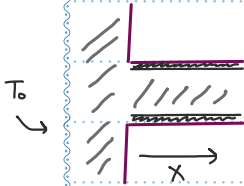
$$\frac{dy}{dx} + f(x)y = g(x) \rightarrow \text{ODE}$$

one independent variable

$$\frac{\partial y}{\partial t} = \alpha \frac{\partial y}{\partial x} \rightarrow \text{PDE}$$

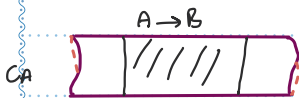
two or more independent variable

→ heat flux in (fin)



$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial C_0}{\partial t} = D_{AB} \frac{\partial^2 C_A}{\partial x^2}$$



$$\frac{\partial C_A}{\partial t} = D_{AB} \frac{\partial^2 C_A}{\partial x^2} + v \frac{\partial C_A}{\partial x} + K C_A$$

$$a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + d \frac{\partial u}{\partial x} + e \frac{\partial u}{\partial y} + f u = g.$$

a, b, c, d, e, f, g → function of (x, y)

example :

$$3x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} + (1 + 3e^y) \frac{\partial^2 u}{\partial y^2} = \cos(x, y).$$

PDE → linear → $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = x^3$

→ non linear → $\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial y}\right)^2 = 0$, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u^2 = 0$

order of PDE

first order → $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

2nd order → $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} = 0$

higher order → $\frac{\partial^n u}{\partial x^n} + \dots = \dots$



$$b^2 - 4ac = ?$$

Case (1) :

$$b^2 - 4ac < 0 \quad \text{elliptic PDE}$$

ex :

$$\underbrace{\frac{\partial^2 u}{\partial x^2}}_a + \underbrace{\frac{\partial^2 u}{\partial x \partial y}}_b + \underbrace{\frac{\partial^2 u}{\partial y^2}}_c = x \frac{\partial u}{\partial x}$$

$$a \quad b \quad c$$

$$a=1, b=1, c=1$$

$$b^2 - 4ac = (1)^2 - 4(1)(1) = -3 < 0$$

Case (2) :

$$b^2 - 4ac > 0 \quad \text{hyperbolic PDE}$$

ex :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = x \frac{\partial u}{\partial x}$$

$$a=1, b=1, c=-1$$

$$b^2 - 4ac = (1)^2 - 4(1)(-1) = +5 > 0$$

Case (3) :

$$b^2 - 4ac = 0 \rightarrow \text{Parabolic PDE}$$

ex :

$$\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = x \frac{\partial u}{\partial x}$$

$$a=1, b=2, c=1$$

$$b^2 - 4ac = (2)^2 - 4(1)(1) = 0$$

example :

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \rightarrow \alpha \frac{\partial^2 T}{\partial x^2} - \frac{\partial T}{\partial t} = 0$$

$$a=\alpha, b=0, c=0$$

$$b^2 - 4ac = 0$$

1) separation of variables

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial y^2} \rightarrow C = f(t) g(y)$$

$$\frac{\partial C}{\partial t} = g(y) \frac{df(t)}{dt} = g f'$$

$$|C, \text{ at } t=0 \rightarrow CA = CA_0$$

$$|C, \text{ at } y=0, CA = CA_0 \quad \frac{\partial C}{\partial y} = f(t) \frac{dg}{dy}$$

$$\text{at } y=l, CA = CA_l$$

$$\frac{\partial^2 C}{\partial y^2} = f(t) \frac{d^2 g}{dy^2}$$

$$g f' = f g'' \rightarrow \frac{f'}{f} = \frac{g''}{g} \rightarrow \frac{1}{f} \frac{df}{dt} = \frac{1}{g} \frac{d^2 g}{dy^2}$$

$$\frac{1}{f} \frac{df}{dt} = \frac{1}{g} \frac{d^2 g}{dy^2} = \text{constant} = \lambda^2$$

$$\frac{1}{f} \cdot \frac{df}{dt} = \lambda^2 \rightarrow \int \frac{1}{f} df = \int \lambda^2 dt \rightarrow \ln f = \lambda^2 t + C \rightarrow f = \alpha e^{\lambda^2 t}$$

$$T(t, x) = T(t) X(x)$$

$$= ce^{-\lambda^2 t} [A \cos \lambda x + B \sin \lambda x]$$

$$* \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \begin{cases} \text{ic (initial condition) } T(0, x) = 3 \sin 2x \\ \text{BC}_1 \text{ (boundary condition 1) } T(t, 0) = 0 \\ \text{BC}_2 \text{ } T(t, 1) = 0 \end{cases}$$

$$\rightarrow T(t, x) = [A' \cos \lambda x + B' \sin \lambda x] e^{-\lambda^2 t}$$

BC₁:

$$0 = [A' \cos \lambda(0) + B' \sin \lambda(0)] e^{-\lambda^2 t}$$

$$A' \cos \lambda(0) + B' \sin \lambda(0) = 0$$

$A' \cos \lambda(0) = 0 \rightarrow \cos(0) = 1$ then A should be zero

$$A' = 0$$

$$\rightarrow T(t, x) = B' \sin \lambda x e^{-\lambda^2 t}$$

BC₂:

$$0 = B' \sin \lambda(1) e^{-\lambda^2 t}$$

$B' \sin \lambda = 0$ B' should not be zero because it becomes trivial solution

$$\sin \lambda = 0$$

$$\lambda = 0, \pi, 2\pi, 3\pi = n\pi, n = 0, 1, 2, \dots$$

$$\rightarrow T(t, x) = B' \sin(n\pi x) e^{-n^2 \pi^2 t}$$

IC:

$$3 \sin 2x = B' \sin(n\pi x) e^0$$

$$3 \sin 2x = B' \sin(n\pi x)$$

$$B' = 3, n\pi = 2 \rightarrow n = \frac{2}{\pi}$$

$$\rightarrow T(t, x) = 3 \sin\left(\frac{2}{\pi} \cdot \pi x\right) e^{-\frac{4}{\pi^2} \pi^2 t}$$

$$T(t, x) = 3 \sin(2x) e^{-4t}$$

$$\text{Ex. } \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \begin{cases} \text{ic } T(0, x) = 2 \sin 3x + 3 \cos 2x \\ T(t, 0) = 0 \\ \text{BC}_2 \text{ } T(t, 1) = 0 \end{cases}$$

$$T(t, x) = T(t) X(x)$$

$$= ce^{-\lambda^2 t} [A \cos \lambda x + B \sin \lambda x]$$

$$= A' \cos \lambda x + B' \sin \lambda x e^{-\lambda^2 t}$$

$$A' = 0, \sin \lambda = 0 \rightarrow \lambda = n\pi$$

$$\rightarrow T(t, x) = B' \sin(n\pi x) e^{-n^2 \pi^2 t}$$

ic:

$$5 \sin 3x + 3 \cos 2x = B' \sin(n\pi x) + 0$$

by Fourier series, convert the 1st term to new sin, cos terms then $\sin = \sin$ $\cos = 0$



121 laplace

$$\text{ex: } \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

$$\text{ic } T(0, x) = 3 \sin 2x \rightarrow t=0$$

$$\text{bc}_1: T(t, 0) = 0 \quad \text{bc}_2: T(t, 1) = 0$$

$$\mathcal{L} [dT/dt] = sT(s, x) - T(t=0)$$

$$\mathcal{L} [d^2T/dx^2] = \frac{d^2T(s, x)}{dx^2}$$

$$\mathcal{L} [3 \sin 2x] = 3 \sin 2x / s$$

$$\rightarrow sT(s, x) - T(0) = \frac{d^2T(s)}{dx^2}$$

$$sT - \frac{3 \sin 2x}{s} = \frac{d^2T}{dx^2}$$

$$\frac{d^2T}{dx^2} - sT = -\frac{3 \sin 2x}{s}$$

$$T = f(x, s)$$

ex:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \quad T(t=0) = 0$$

$$sT(s) - \overset{0}{T(0)} = \frac{d^2T}{dx^2}$$

$$\frac{d^2T}{dx^2} - sT(s) = 0$$

$$T = e^{rx}, \quad T' = r e^{rx}, \quad T'' = r^2 e^{rx}$$

$$\rightarrow r^2 e^{rx} - s e^{rx} = 0$$

$$r^2 - s = 0, \quad r = \pm \sqrt{s}$$

$$\rightarrow T = A e^{\sqrt{s}x} + B e^{-\sqrt{s}x}$$

$$\neq \mathcal{L}^{-1}[e^{-\sqrt{s}}] = \frac{a}{2\sqrt{\pi t^3}} e^{-\frac{1}{4}t^2 x}$$

$$\mathcal{L}^{-1}[e^{\sqrt{s}}] = \infty, \quad T = A(\infty) + B a e^{-\frac{1}{4}t^2 x} / 2\sqrt{\pi t^3}$$