Conduction with thermal energy generation

The Plane Wall

&

The Solid Cylinder

Heat Diffusion Equation- Other forms

- If k=constant

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$\alpha=\dfrac{k}{\rho c_p}$ is the thermal diffusivity

(2.3)

- For steady state conditions

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=0$$

(2.4)

- For steady state conditions, one-dimensional transfer in x-direction and no energy generation

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0 \quad \text{or} \quad \frac{dq''_x}{dx}=0$$

➢ Heat flux is constant in the direction of transfer

The Plane Wall

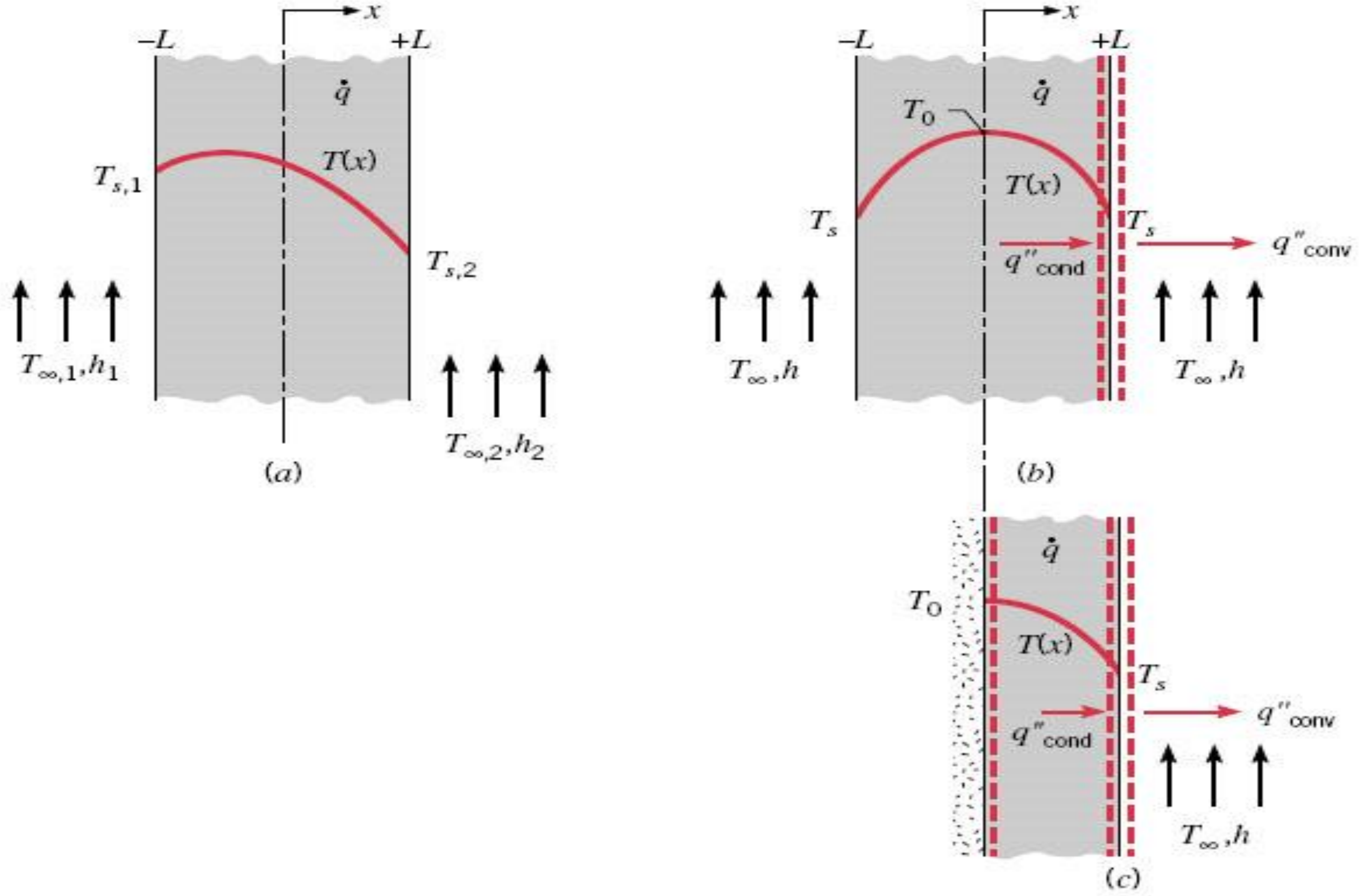


FIGURE 3.9 Conduction in a plane wall with uniform heat generation. (*a*) Asymmetrical boundary conditions. (*b*) Symmetrical boundary conditions. (*c*) Adiabatic surface at midplane.

Temp distribution through the plane wall with energy generation

Consider the plane wall of Figure 3.9*a*, in which there is *uniform* energy generation per unit volume ($\dot{q}$ is constant) and the surfaces are maintained at $T_{s,1}$ and $T_{s,2}$. For constant thermal conductivity k, the appropriate form of the heat equation, Equation 2.20, is

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \qquad (3.39)$$

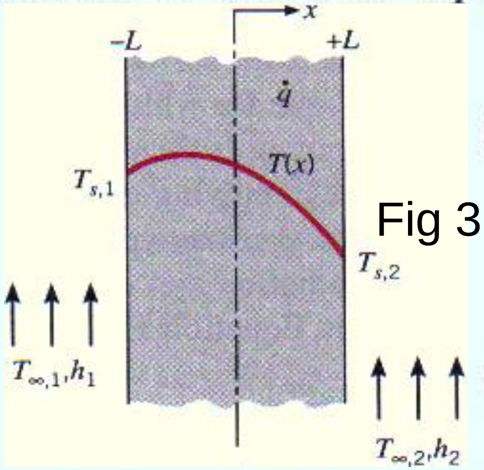


Fig 3.9a

The general solution is

$$T = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2 \qquad (3.40)$$

where C_1 and C_2 are the constants of integration. For the prescribed boundary conditions,

$$T(-L) = T_{s,1} \qquad \text{and} \qquad T(L) = T_{s,2}$$

The constants may be evaluated and are of the form

$$C_1 = \frac{T_{s,2} - T_{s,1}}{2L} \qquad \text{and} \qquad C_2 = \frac{\dot{q}}{2k}L^2 + \frac{T_{s,1} + T_{s,2}}{2}$$

in which case the temperature distribution is

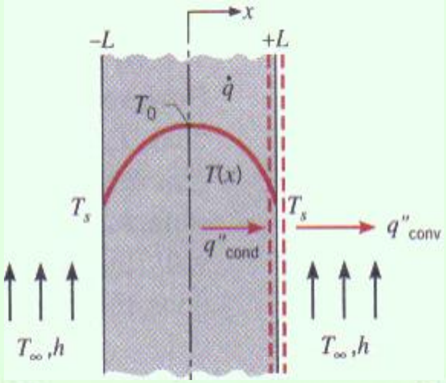
$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2} \qquad (3.41)$$

The heat flux at any point in the wall may, of course, be determined by using Equation 3.41 with Fourier's law. Note, however, that *with generation the heat flux is no longer independent of x.*

For T_{s1}=T_{s2}=Ts ; hence

Eq. 3.41 becomes

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s \qquad (3.42)$$



The maximum temperature exists at the midplane

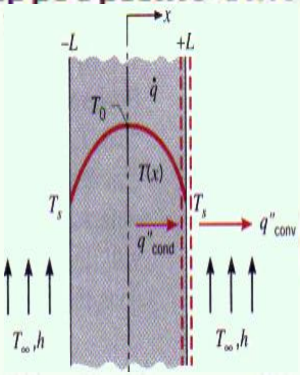
$$T(0) \equiv T_0 = \frac{\dot{q}L^2}{2k} + T_s \qquad (3.43)$$

in which case the temperature distribution, Equation 3.42, may be expressed as

$$\frac{T(x) - T_0}{T_s - T_0} = \left(\frac{x}{L}\right)^2 \qquad (3.44)$$

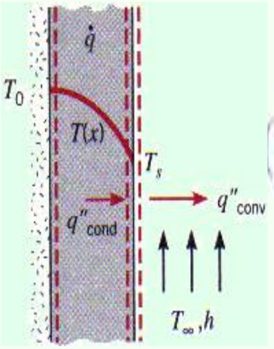
Common case T_∞ given not T_s

for surface at $x = L$ for the symmetrical plane wall (Figure 3.9*b*) or the insulated plane wall (Figure 3.9*c*). Neglecting radiation and substituting the appropriate rate equations, the energy balance given by Equation 1.12 reduces to



$$-k\frac{dT}{dx}\bigg|_{x=L} = h(T_s - T_\infty) \qquad (3.45)$$

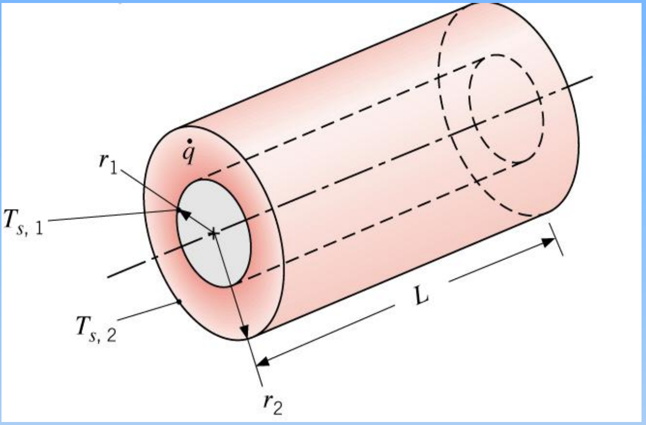
Substituting from Equation 3.42 to obtain the temperature gradient at $x = L$, it follows that

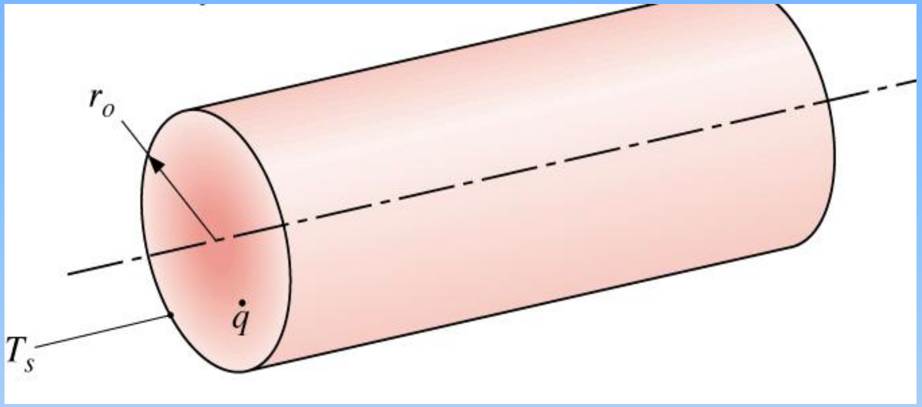


$$T_s = T_\infty + \frac{\dot{q}L}{h} \qquad (3.46)$$

Hence T_s may be computed from knowledge of T_∞, $\dot{q}$, L, and h.

Types of cylinders

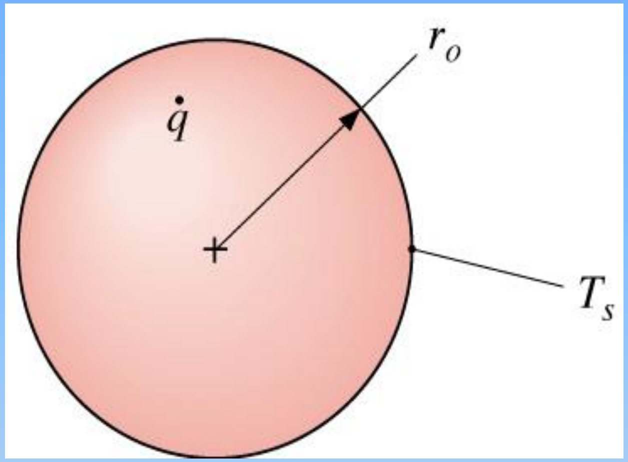




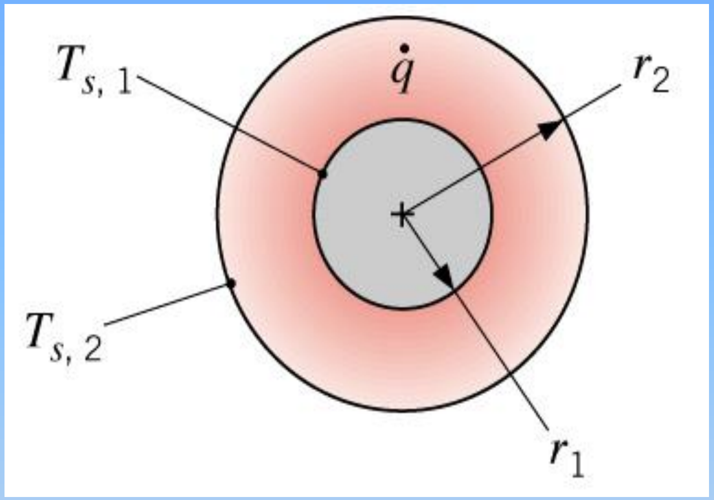
Cylindrical (Tube) Wall

Solid Cylinder (Circular Rod)

Types of spheres



Solid Sphere



Spherical Wall (Shell)

The Solid Cylinder

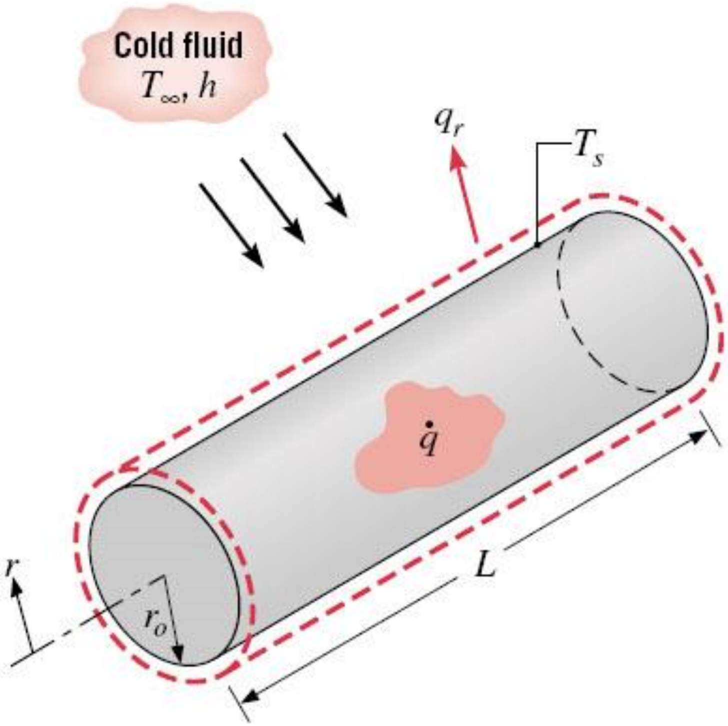


FIGURE 3.10
Conduction in a solid cylinder with uniform heat generation.

Heat Diffusion Equation

- In cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \qquad (2.5)$$

- In spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

(2.6)

Temp distribution through solid cylinder with uniform energy generation

To determine the temperature distribution in the cylinder, we begin with the appropriate form of the heat equation. For constant thermal conductivity k, Equation 2.24 reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0 \qquad (3.49)$$

Separating variables and assuming uniform generation, this expression may be integrated to obtain

$$r\frac{dT}{dr}=-\frac{\dot{q}}{2k}r^2+C_1 \qquad (3.50)$$

Repeating the procedure, the general solution for the temperature distribution becomes

$$T(r)=-\frac{\dot{q}}{4k}r^2+C_1\ln r+C_2 \qquad (3.51)$$

To obtain the constants of integration C_1 and C_2, we apply the boundary conditions

$$\left.\frac{dT}{dr}\right|_{r=0}=0 \qquad \text{and} \qquad T(r_o)=T_s$$

at $r = 0$ and Equation 3.50, it is evident that $C_1 = 0$. Using the surface boundary condition at $r = r_o$ with Equation 3.51, we then obtain

$$C_2 = T_s + \frac{\dot{q}}{4k} r_o^2 \qquad (3.52)$$

The temperature distribution is therefore

$$T(r) = \frac{\dot{q} r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right) + T_s \qquad (3.53)$$

Evaluating Equation 3.53 at the centerline and dividing the result into Equation 3.53, we obtain the temperature distribution in nondimensional form,

$$\frac{T(r) - T_s}{T_o - T_s} = 1 - \left(\frac{r}{r_o}\right)^2 \qquad (3.54)$$

Relation between T_s and T_∞

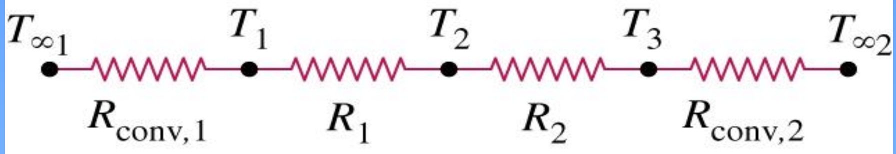
To relate the surface temperature, T_s, to the temperature of the cold fluid, T_∞, either a surface energy balance or an overall energy balance may be used. Choosing the second approach, we obtain

$$\dot{q}(\pi r_o^2 L) = h(2\pi r_o L)(T_s - T_\infty)$$

or

$$T_s = T_\infty + \frac{\dot{q} r_o}{2h} \qquad (3.55)$$

Thermal resistance network for a cylinder



$$\dot{Q} = \frac{T_{\infty 1} - T_1}{R_{conv,1}}$$

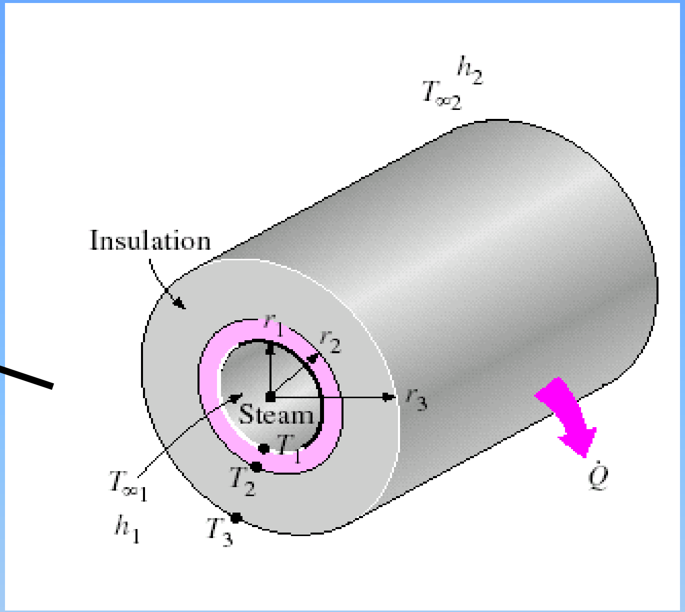
$$= \frac{T_{\infty 1} - T_2}{R_{conv,1} + R_1}$$

$$= \frac{T_1 - T_3}{R_1 + R_2}$$

$$= \frac{T_2 - T_3}{R_2}$$

$$= \frac{T_2 - T_{\infty 2}}{R_2 + R_{conv,2}}$$

$$= \cdots$$



$$R_{total} = R_{conv,1} + R_{cyl} + R_{conv,2}$$

$$= \frac{1}{(2\pi r_1 L)h_1} + \frac{\ln(r_2/r_1)}{2\pi L k} + \frac{1}{(2\pi r_2 L)h_2}$$